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EXISTENCE FOR THE LINEARIZATION OF A STEADY STATE FLUID/NONLINEAR ELASTICITY INTERACTION

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Abstract. A linearized steady state three-dimensional fluid-structure interaction is considered and its solvability is studied. The linearization (obtained in a previous work by these authors) that we deal with has new features, including the presence of the curvature terms on the common interface. These new extra terms, coming from the geometrical aspect of the problem, are critical for a correct physical interpretation of the fluid/structure coupling. We prove that the linearization has unique solution.

1. Introduction.

1.1. The problem and the model.

The stability of free boundary problems is relevant to many modern issues, from medicine to nuclear power. The classical approach to stability analysis is to consider the linearization of a nonlinear system around the physical parameters of its "working" configuration, to ensure that the model is relevant to the real functionality of the system. For example, the stability of plane wings is analyzed when the plane is at high speed, carrying a substantial weight, not when empty of cargo and at rest. Alternatively, the stability of a submarine can be done with the submarine moving or at rest, but in either case the water pressure must be non-zero.

We are interested in the problem of stabilization for the coupling of viscous fluid and nonlinear elasticity; such models describe, for example, the dynamics of blood flow inside arteries (with many applications in the study of arterial diseases). In any coupling of fluid and structure, such as those occurring in arteries, submarines, and bridges, a challenge arises in just *how* to linearize the free boundary. Previous results have accomplished this by coupling the linearizations of each system separately.

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However, our new, alternative approach [7] is to couple the systems *first*, yielding a complete non-linear model which can then be linearized. This method is more challenging mathematically, resulting in a free boundary which moves with the given perturbation parameter. Adapting recent results developed in shape optimization and image analysis [8, 9, 14, 16, 18, 19, 21] allowed us to fully characterize the linearization of the overall coupled system, in the case of a *steady-state* fluid / nonlinear elasticity interaction [7]. Our new approach can be briefly summarized as follows: we started with the steady state interaction between an incompressible, viscous fluid (modeled by the Navier-Stokes equation) and a 3-d nonlinear elastic body, which took place on the common interface and was accomplished via suitable transmission boundary conditions; we assumed existence of the fluid-structure interaction, and we derived the linearization of the system by perturbing the steady regime using a parameter of variation s, and then computing the shape derivatives with respect to s at $s = 0$ [18]; in the end, we obtained the linearization of the coupled fluid-structure problem around fluid rest, which revealed new features, including the presence of the curvature terms on the common boundary, captured in the matrix of curvatures D^2b, where b is the oriented distance function; see [14, 15] for a proper definition and properties of the oriented distance function.

Our approach highlights the geometrical aspects of the problem, like curvature and boundary acceleration, which are unaccounted for in previous models. This is particularly important in the case when the boundary oscillates, sending the mean curvature to infinity. Modeling of this geometrical aspect is critical for a correct physical interpretation of the fluid-structure interaction.

Our goal in this paper (which can be seen as a continuation of [7]) is to analyze the linearized steady system obtained in [7] and described above, as a first step towards the problem of stabilization (which will use time depending existence results [10, 12, 17] and techniques present in [1, 2, 3, 4, 5, 6]). The presence of the extra terms (in particular the one involving the matrix of curvatures) in a Fourier (Robin)-type condition on the boundary of the linearized coupling of nonlinear elasticity and fluid also makes the problem of existence and uniqueness of solutions **new** and challenging. The novelty of the present paper comes from the fact that it provides a first result of existence and uniqueness of solutions to a linearized coupled system where the boundary and its curvatures are not neglected, like in usual couplings of linear elasticity and fluid. We are able to provide technical estimates to deal with the "delicate terms" (which appear as tangential derivatives in a Neumann boundary condition type), and a good setting for the problem where we can use a variational approach to prove existence and uniqueness of solution.

The rest of the paper is organized as follows. In Section 1.2, we introduce the notation that will be used throughout the paper. Section 1.3 presents the linearized PDE model, as well as a brief description of the technique used for the linearization. Finally, Section 2 contains the existence and uniqueness result, as well as its proof.

1.2. **Notation.** For the rest of the paper, we use the repeated index convention for summation whenever the same Latin index appears twice, and the following notation:

•$(Df(a))_{ij} = \partial_j f_i(a) \in \mathbb{M}^3$ is the gradient matrix at $a \in X$ of any vector field $f = (f_i) : X \subset \mathbb{R}^3 \to \mathbb{R}^3$.

•$\mathrm{div} f(a) = \partial_i f_i \in \mathbb{R}$ is the divergence of $f : X \subset \mathbb{R}^3 \to \mathbb{R}^3$ at $a \in X$.

•$\mathrm{Div}\, T(a) = \partial_j T_{ij} e_i \in \mathbb{R}^3$ is the divergence of any second-order tensor field $T = (T_{ij}) : X \subset \mathbb{R}^3 \to \mathbb{M}^3$ at $a \in X$.

Since we ignore the distinction between covariant and contravariant components, we will identify the set of all second-order tensors with the set $\mathbb{M}^3$ of all square matrices of order three.

•$A^* =$ transpose of A, for any $A \in \mathbb{M}^3$.
•$A^{-*} = \left(A^{-1}\right)^*$
•$A..B = \mathrm{Tr}(A^*B) \in \mathbb{R}$ is the matrix inner product in $\mathbb{M}^3$.
•$\mathrm{Cof}(A) = \det(A)A^{-*}$ is the cofactor matrix of any invertible matrix $A \in \mathbb{M}^3$.
•$d_\Omega(x) = \begin{cases} \inf_{y\in\Omega} |y-x| & , \Omega \neq \emptyset \\ \infty & , \Omega = \emptyset \end{cases}$ is the distance function from x to $\Omega \in \mathbb{R}^n$.
•$b_\Omega(x) = d_\Omega(x) - d_{\Omega^c}(x)$, $\forall x \in \mathbb{R}^n$ is the oriented distance function from x to Ω, for any $\Omega \subset \mathbb{R}^n$.
•$H = \mathrm{Tr}(D^2 b_\Omega)|_\Gamma = (\Delta b_\Omega)|_\Gamma$ is the mean curvature of Γ.

1.3. **PDE Model.** We now describe the initial nonlinear PDE model under consideration and the linearization around fluid rest, obtained in [7]. Let $\mathcal{D} \in \mathbb{R}^3$ be a bounded domain comprised of two open domains $\mathcal{D} = \Omega \cup \Omega^c$, and with Lipschitz continuous boundary (piecewise smooth boundary) $\partial\mathcal{D} = \Gamma' \cup \Gamma_{in} \cup \Gamma_{out}$ (Γ', Γ_{in}, and Γ_{out} are closed); see the figure below.

The elastic body occupies a domain Ω with locally Lipschitz boundary $\Gamma \cup \Gamma'$, and is described by a three-dimensional nonlinear Saint Venant elastic equation (large displacement, small deformation elasticity) in terms of the displacement u. The fluid occupies domain Ω^c with boundary $\Gamma \cup \Gamma_{in} \cup \Gamma_{out}$, and is described by a Navier-Stokes equation in terms of the velocity of the fluid w and the pressure p. We consider the viscosity of the fluid $\nu = 1$. The fluid sticks to the boundary Γ, which corresponds to a homogeneous boundary condition. The interaction takes place on the common boundary Γ and is realized via suitable transmission boundary conditions: we require continuity of both the velocities (the velocity of the fluid and the velocity of the boundary) and the normal stress tensors across the interface Γ. We assume that there is a flux $\vec{f}$ coming into $\mathcal{D}$ through Γ_{in} (described below in (1.3)), that will determine the velocity of the fluid w.

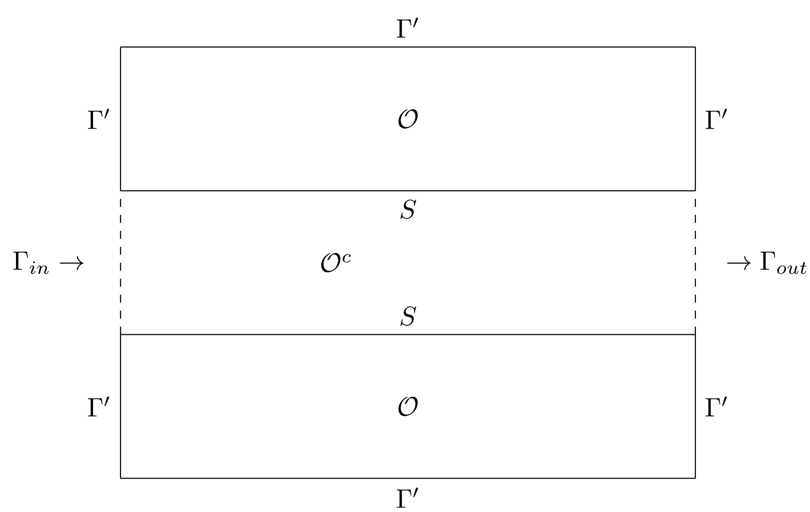

Figure 1: Reference configuration (at rest); 2D axial cross-section

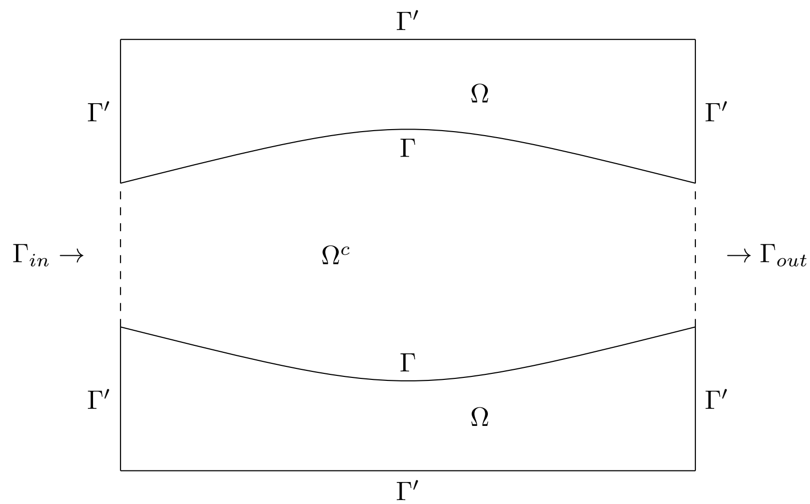

Figure 2: Deformed configuration; 2D axial cross-section

The PDE model for the fluid-structure interaction defined by the variables (w, p, u) is given by

$$\begin{cases} -\Delta w + Dw \cdot w + \nabla p = 0 & \Omega^c \\ \text{div } w = 0 & \Omega^c \\ -\text{Div } \mathcal{T} = 0 & \Omega \\ w = g & \partial\Omega^c = \Gamma \cup \Gamma_{in} \cup \Gamma_{out} \\ \mathcal{T} n = pn - 2\varepsilon(w)n & \Gamma \\ u = 0 & \Gamma' \end{cases} \tag{1.1}$$

The Dirichlet data $g \in H^{1/2}(\partial\Omega^c)$ are given, $2\varepsilon(w) = Dw + Dw^*$ is the linear strain tensor matrix, and $\mathcal{T} : \bar{\Omega} \to \mathbb{S}^3$ is the Cauchy stress tensor given by:

$$\mathcal{T} = \Big(\frac{1}{\det(D\varphi)} D\varphi \cdot \Sigma(\sigma(u)) \cdot (D\varphi)^* \Big) \circ \varphi^{-1} \tag{1.2}$$

Where $\varphi = I + u$ is the deformation of the reference configuration $\overline{\mathcal{O}} \in \mathbb{R}^3 \to \mathbb{R}^3$, $\sigma(u) = \frac{1}{2}(Du^* + Du + (Du)^* Du)$ is the Green-St Venant strain tensor, and $\Sigma(\sigma(u)) = \lambda(\text{Tr}\sigma(u))I + 2\mu\sigma(u)$ defines the second Piola-Kirchhoff stress tensor, with λ and μ being the Lamé constants of the material [11].

Note that we have to use the elasticity equations on the deformed configuration (Ω, Γ) (instead of the reference configuration $(\mathcal{O}, \mathcal{S})$), since we are dealing with an elastic body/fluid system, where the coupling is taking place on the boundary interface between the two media. This interface is precisely the boundary Γ of the deformed configuration of the elastic body Ω and thus the coupling requires the continuity of the velocities and the normal stress tensors across Γ. This explains the relationship between the Cauchy stress tensor $\mathcal{T}$ and the strain tensor $\sigma(u)$, which will provide us with the correct matching of the two dynamics on the common interface. A detailed explanation on the nonlinear elastic component of the coupled system and formula (1.2) is given in Appendix A [7].

With (1.1), we associate the following example for the function $g \in H^{1/2}(\partial\Omega^c)$: (i) on Γ, the fluid sticks to the boundary, hence $g = 0$ on Γ, and (ii) the boundary conditions on Γ_{in} and Γ_{out} are described as follows: let $c_{in}(x)$ (respectively $c_{out}(x)$)

be a given, smooth function defined on Γ_{in} (respectively Γ_{out}) such that

$$\begin{cases} c_{in}(x) = 0 & \text{on } \partial\Gamma_{in}, \\ w = c_{in}(x)\, n_{in} & \text{on } \Gamma_{in} \end{cases} \tag{1.3}$$

(i.e., $g = c_{in}(x)\, n_{in}$ on Γ_{in}), where n_{in} is the unit outer normal vector along Γ_{in}, and a similar condition on Γ_{out}:

$$\begin{cases} c_{out}(x) = 0 & \text{on } \partial\Gamma_{out}, \\ w = \alpha\ c_{out}(x)\, n_{out} & \text{on } \Gamma_{out} \end{cases} \tag{1.4}$$

(i.e., $g = \alpha\ c_{out}(x)\, n_{out}$ on Γ_{out}), where n_{out} is the unit outer normal vector along Γ_{out}, and $\alpha \in \mathbb{R}$ is chosen in order to verify the compatibility condition with respect to the divergence free property of the fluid:

$$\alpha = -\left(\int_{\Gamma_{out}} c_{out}(x)\ d\Gamma_{out}\right)^{-1} \int_{\Gamma_{in}} c_{in}(x)\ d\Gamma_{in} \tag{1.5}$$

Parameter of Variation s: We perturb the steady regime presented above by assuming that the flux entering the domain $\mathcal{D}$ is dependent on a variation parameter s, i.e., $c_{in}(x)$ (respectively $c_{out}(x)$) is a given, smooth function defined on Γ_{in} (respectively Γ_{out}) such that for some constant $a \geq 0$,

$$\begin{cases} c_{in}(x) = 0 & \text{on } \partial\Gamma_{in}, \\ w_s = (a+s)c_{in}(x)\, n_{in} & \text{on } \Gamma_{in}, \end{cases} \tag{1.6}$$

and for any $s \geq 0$, we choose $\alpha_s \in \mathbb{R}$ verifying

$$\begin{cases} c_{out}(x) = 0 \text{ on } \partial\Gamma_{out}, \\ w_s = \alpha_s c_{out}(x)\, n_{out} \text{ on } \Gamma_{out}, \\ \alpha_s = -(a+s)\int_{\Gamma_{in}} c_{in}(x)\ d\Gamma_{in}\ (\int_{\Gamma_{out}} c_{out}(x)\ d\Gamma_{out})^{-1}, \text{ for all } s \geq 0 \end{cases} \tag{1.7}$$

If the elastic body occupies a reference configuration $\overline{\mathcal{O}} \in \mathbb{R}^3$ with Lipschitz boundary $\mathcal{S} \cup \Gamma'$, then, when subjected to applied forces, it occupies a deformed configuration $\Omega_s = \varphi_s(\overline{\mathcal{O}})$, with Lipschitz boundary $\Gamma_s \cup \Gamma'$ (where Γ' is fixed). The deformation map in this case is dependent on the parameter s: $\varphi_s : \overline{\mathcal{O}} \to \mathbb{R}^3$, but nevertheless is smooth enough, injective, and orientation-preserving. The displacement $u_s : \overline{\mathcal{O}} \to \mathbb{R}^3$ becomes $u_s = \varphi_s - I$, where I is the identity map $I : \overline{\mathcal{O}} \to \mathbb{R}^3$. Similarly, for the fluid present in the system, the velocity and pressure are now functions of s: w_s, and p_s, and thus we have the following coupled system for the interaction:

$$\begin{cases} -\Delta w_s + Dw_s \cdot w_s + \nabla p_s = 0 & \Omega_s^c \\ \text{div } w_s = 0 & \Omega_s^c \\ -\text{Div } \mathcal{T}_s = 0 & \Omega_s \\ w_s = 0 & \Gamma_s \\ w_s = (a+s)c_{in}(x)\, n_{in} & \Gamma_{in} \\ w_s = \alpha_s c_{out}(x)\, n_{out} & \Gamma_{out} \\ \mathcal{T}_s n_s = p_s n_s - 2\varepsilon(w_s) n_s & \Gamma_s \\ u_s = 0 & \Gamma' \end{cases} \tag{1.8}$$

where $c_{in}(x)$ and $c_{out}(x)$ were introduced in (1.6) and (1.7), n_s is the unit outer normal vector along Γ_s, $2\varepsilon(w_s) = Dw_s + Dw_s^*$, and $\mathcal{T}_s : \bar{\Omega}_s \to \mathbb{S}^3$ is the Cauchy stress tensor (associated to s), given by

$$\mathcal{T}_s = \Big(\frac{1}{\det(\nabla\varphi_s)}\nabla\varphi_s \cdot \Sigma(\sigma(u_s)) \cdot (\nabla\varphi_s)^*\Big) \circ \varphi_s^{-1}. \tag{1.9}$$

Below we present the result that describes the linearization of the Navier-Stokes fluid/elastic structure. Let $w' = \frac{\partial}{\partial s}w_s\Big|_{s=0}$, $p' = \frac{\partial}{\partial s}p_s\Big|_{s=0}$, and $u' = \frac{\partial}{\partial s}u_s\Big|_{s=0}$ be the shape derivatives of (w, p, u) with respect to s at $s = 0$. The linearization was obtained around "fluid rest", i.e., for $w = 0$. Note that if in (1.8) we assume that $a = 0$, then at $s = 0$ we obtain $w = 0$.

Theorem 1.1 (Linearization around fluid rest for the coupling viscous fluid - elastic structure [7]). *In system (1.8), we assume that $a = 0$. Then we have the following linearized model around fluid at rest:*

$$\begin{cases} -\Delta w' + \nabla p' = 0 & \Omega^c \\ div\, w' = 0 & \Omega^c \\ w' = 0 & \Gamma \\ -\vec{D}iv(\mathcal{T}') = 0 & \Omega \\ \mathcal{T}'n = (p'I - 2\varepsilon(w'))n + \nabla p \cdot n(U' \cdot n)n & \\ \qquad +(pI - \mathcal{T})(D^*_\Gamma U' \cdot n + (D^2 b_\Omega)U'_\Gamma) - U' \cdot n Div_\Gamma(\mathcal{T}) & \Gamma \\ U' = 0 & \Gamma' \\ w' = c_{in}(x)n_{in} & \Gamma_{in} \\ w' = \alpha c_{out}n_{out} & \Gamma_{out} \end{cases} \tag{1.10}$$

where $U' = u' \circ (I+u)^{-1}$, $\mathcal{T}$ is given by (1.2), and $\mathcal{T}'$ has the following expression:

$$\begin{aligned} \mathcal{T}' = & \Big(- \frac{div(u')}{det(I+Du)}\,(I+Du)[C_{\lambda,\mu}..(\sigma(u))](I+D^*u)\Big) \circ (I+u)^{-1} \\ & + \Big(\frac{1}{det(I+Du)}D(u')[C_{\lambda,\mu}..(\sigma(u))](I+D^*u)\Big) \circ (I+u)^{-1} \\ & + \Big(\frac{1}{det(I+Du)}(I+Du)[C_{\lambda,\mu}..(\sigma')](I+D^*u)\Big) \circ (I+u)^{-1} \\ & + \Big(\frac{1}{det(I+Du)}(I+Du)[C_{\lambda,\mu}..(\sigma(u))]D^*(u')\Big) \circ (I+u)^{-1} \\ & - \mathbf{D}\mathcal{T}(u' \circ (I+u)^{-1}) \end{aligned} \tag{1.11}$$

Remark 1.1. In the above expression, we assumed that the four entries of the elasticity tensor $C_{\lambda,\mu}$ are governed by the *Lamé* coefficients λ and μ, and $C_{\lambda,\mu}..(\sigma(u))$ is a simplified notation for $\lambda \mathrm{Tr}(\sigma(u))I + 2\mu\sigma(u)$. Moreover, $\mathbf{D}\mathcal{T}$ is a three entries tensor, representing the gradient of the matrix $\mathcal{T}$.

Remark 1.2. Note that the coupling obtained in (1.10) is more complicated than just the usual coupling of the linear problems in the variables (u', w', p'). Indeed, the boundary curvatures play an important role in the analysis of the coupled fluid-structure interaction. These terms can not be neglected [13], since when the boundary has oscillations, the mean curvature $H = \mathrm{Tr}(D^2 b_\Omega)|_\Gamma$ is not bounded.

Remark 1.3. Due to the fact that we have $(\mathcal{T} - pI)n = 0$ *on* Γ, the boundary condition in (1.10) can be simplified as follows:

$$(pI - \mathcal{T})(D^*_\Gamma U')n \,=\, (pI - \mathcal{T})(D^* U')n \;\; on\ \Gamma$$

2. **Existence for the linearization of a steady state fluid / nonlinear elasticity interaction.** In this section we provide our result on existence of a unique solution (w', p', U') for the coupled PDE system (1.10). As mentioned before, the main difficulty of the problem is represented by the presence of the complicated coupling on the common interface Γ. There are four new, extra terms on Γ (involving tangential gradient of the main variable U', and the matrix of curvatures) that need to be dealt with, in comparison to the usual coupling of linearized equation, where the condition simplifies to the usual matching of the normal stress tensors across the common interface.

We introduce the following set:

$$B_\rho = \{(u,p) \in C^2(\bar{\mathcal{O}}) \times C^1(\bar{\Omega}^c) \mid ||u||_{C^2(\bar{\mathcal{O}})} \leq \rho,\, ||q||_{C^1(\bar{\mathcal{O}}^c)} \leq \rho\}$$

where $q = p \circ (I + \bar{u})$, with $\bar{u} \in C^2(D)$ being any extension of $u \in C^2(\mathcal{O})$ to the whole domain D; for the construction of a specific extension, see (2.7).

Now we can state our existence and uniqueness theorem:

Theorem 2.1. *There exists $\rho^* > 0$ such that $\forall \rho < \rho^*$ and $(u,p) \in B_\rho$, solution to (1.1) with $w = 0$, there exists unique solution $(w', p', U') \in H^1(\Omega^c) \times L^2(\Omega^c) \times H^1_{\Gamma'}(\Omega)$ to (1.10).*

In the remainder of the section we will provide the proof of Theorem 2.1. These are the main steps of the proof:

1. We take advantage of the fact that the linearized problem (1.10) is weakly coupled and first solve the Stokes equation in variables (w', p').
2. The linear elasticity problem is characterized by a non-standard boundary condition of Fourier-like type, where w' and p' appear as data. The goal is to write the problem in variational form. In order to do that, we introduce a non-symmetric bilinear form $A^1_{u,p}$, along with the correct functional framework for its definition (due to the lack of smoothness of the domain, the regularity obtained in the previous step for (w', p') does not automatically imply the desired L_2 regularity for the boundary data present in the elasticity system, i.e., $(p'I - 2\epsilon(w')).n$).
3. Now the goal is to apply Lax-Milgram theorem for a non-symmetrical coercive form. The bilinear form $A^1_{u,p}$ is defined on domains and boundaries that move with u, thus we first "bring it back" to the reference configuration $\mathcal{O}$, in order to apply a perturbation argument.
4. We show that the "transported" bilinear form is coercive (making use of some technical estimates) and by Lax-Milgram, we obtain existence of a unique solution to the linearized problem.

Now we provide the details of the proof.

Proof. **The Stokes system**: The coupled problem (1.10) with solution (w', p', U') is weakly coupled, since the linear Stokes equation in the variables (w', p') is well posed and can be directly solved. Moreover, we assume the boundary data on Γ_{in} and Γ_{out} smooth enough to ensure the existence and uniqueness of solution $(w', p') \in H^1(\Omega^c) \times L^2(\Omega^c)$ for the Stokes component of (1.10)[20].

If we consider the closed convex set

$$K = \{ \ \phi \in H^1(\Omega^c), \operatorname{div} \phi = 0, \ \ \phi = g \text{ on } \Gamma_{in} \cup \Gamma_{out}, \ \phi = 0 \text{ on } \Gamma \ \}$$

and the functional

$$J(\phi) = 1/2 \int_{\Omega^c} \varepsilon(\phi)..\varepsilon(\phi)\, dx,$$

it is well known that J reaches its minimum in K at a single element $w' \in K$ characterised by

$$\forall \phi \in K, \ \ J'(w', \phi - w') \geq 0$$

By setting $\theta = w' - \phi \in K - K$ (*which is a linear space*), we have that

$$\forall \theta \in K - K, \ \ J'(w', \theta) = \int_{\Omega^c} \varepsilon(w')..\varepsilon(\theta) dx = 0$$

Now we introduce a larger space than $K - K$. Let

$$H^1(\Omega^c)_{\Gamma_{in} \cup \Gamma_{out}} = \{\theta \in H^1(\Omega^c)), \ \ \theta = 0 \text{ on } \Gamma_{in} \cup \Gamma_{out}\}$$

Using standard integration by parts, equation (1.10), and keeping in mind that θ is not divergence free, we obtain the following identity, $\forall w' \in K$, $\forall \theta \in H^1(\Omega^c)_{\Gamma_{in} \cup \Gamma_{out}}$:

$$\begin{aligned} \int_{\Omega^c} 2\varepsilon(w')..\varepsilon(\theta) dx &= - \int_{\Omega^c} \Delta w' \cdot \theta \, dx \ + \int_{\Gamma} 2\varepsilon(w') n^c \cdot \theta \ d\Gamma \\ &= \int_{\Gamma} (2\varepsilon(w') n^c - p' \, n^c) \cdot \theta \ d\Gamma + \int_{\Omega^c} p' \, div \theta \, dx, \end{aligned} \tag{2.1}$$

where n^c is the unit outer normal vector on Γ with respect to Ω^c.

Now we define the space $H^{1/2}_{00}(\Gamma)$ as follows:

$$H^{1/2}_{00}(\Gamma) = \{\theta \in H^{1/2}(\partial\Omega^c), \ \text{ s.t. } \ \theta = 0 \text{ a.e. in } \ \partial\Omega^c \setminus \Gamma\},$$

for which we have the following lemma:

Lemma 2.1. *The traces of elements of $H^1(\Omega^c)_{\Gamma_{in} \cup \Gamma_{out}}$ describe the entire linear space $H^{1/2}_{00}(\Gamma)$. Moreover, there exists a linear mapping*

$$R_\Gamma \in \mathcal{L}(H^{1/2}_{00}(\Gamma), H^1(\Omega^c)_{\Gamma_{in} \cup \Gamma_{out}}), \ \ s.t. \ \forall \theta \in H^{1/2}_{00}(\Gamma), \ \ (R_\Gamma \theta)|_{\partial\Omega^c} = \theta$$

The lemma is straightforward. If we let $\theta \in H^{1/2}_{00}(\Gamma)$, then we define $\zeta = R_\Gamma \theta$ as the solution of the Laplace problem with Dirichlet boundary condition:

$$\begin{cases} -\Delta \zeta = 0, & \Omega^c \\ \zeta = \tilde{\theta}, & \partial\Omega^c, \end{cases} \tag{2.2}$$

where $\tilde{\theta} \in H^{1/2}(\partial\Omega^c)$ is the extension of θ outside of Γ by 0.

Now we have all the ingredients in order to weakly define the element

$$\tau := 2\varepsilon(w') n^c - p' n^c \in H^{1/2}_{00}(\Gamma)'$$

as follows:

$$\langle 2\varepsilon(w') n^c - p' n^c, \, \theta \rangle_{\{H^{1/2}_{00}(\Gamma)' \times H^{1/2}_{00}(\Gamma)\}} \ = \int_{\Omega^c} [\, 2\varepsilon(w')..\varepsilon(R_\Gamma \theta) \ - \ p' \, div(R_\Gamma \theta) \,] dx \tag{2.3}$$

The elasticity system: Now what it is left to do is to prove existence and uniqueness of the solution $U' \in H^1_{\Gamma'}(\Omega)$ for the elastic component of the coupling (1.10), which can be considered as a linear elliptic problem with Fourier-like boundary condition:

$$\begin{cases} \mathrm{Div}(\mathcal{T}') = 0 & \Omega \\ \mathcal{T}'n - [\,(pI - \mathcal{T})D^*U'\,]n = (p'I\ - 2\varepsilon(w'))n + \nabla p \cdot n\ (U' \cdot n)n & \\ \qquad +(pI - \mathcal{T})(D^2 b_\Omega)U' - (U' \cdot n)\mathrm{Div}_\Gamma(\mathcal{T}) & \Gamma \\ U' = 0 & \Gamma' \end{cases} \tag{2.4}$$

We want to point out that the deformed domains Ω and Γ depend on u, and thus for the rest of the paper we will use Ω_u and Ω interchangeably (same thing for the boundary Γ_u and Γ).

We divide the proof in four main steps:

Step 1: We build a non-symmetrical bilinear form $A^1_{u,p}$ on $H^1_{\Gamma'}(\Omega) = \{f \in H^1(\Omega) \,|\, f = 0 \text{ on } \Gamma'\}$, in order to write (2.4) in variational form.

In the definition of the bilinear form, we will make use of the following lemma and its corollary.

Lemma 2.2.

$$H^1_{\Gamma'}(\Omega) = \{\Theta = \Psi|_\Omega \ \ s.t. \ \ \Psi \in H^1_0(D)\} \tag{2.5}$$

Proof. Let $\Theta \in H^1_{\Gamma'}(\Omega)$. We will show that Θ is the restriction of a function in $H^1_0(D)$ by constructing a specific extension $\Psi \in H^1_0(D)$.

First, we define $\tilde{\Theta} = \Theta \circ (I + u)$. Since $I + u \in C^2(\bar{\mathcal{O}})$ and $u = 0$ on Γ', we have that $\tilde{\Theta} \in H^1_{\Gamma'}(\mathcal{O})$. So now we are back on the reference configuration, where we can take advantage of the symmetry of the domain ($\mathcal{O}$ is an annular cylinder; see Figure 1). Note that we can parametrize $\mathcal{O}$ and $\mathcal{O}^c$ as follows:

$$\mathcal{O} = \{(x, r\cos\alpha, r\sin\alpha), \text{ for } r_0 < r < R_0\} \text{ and}$$
$$\mathcal{O}^c = \{(x, r\cos\alpha, r\sin\alpha), \text{ for } r < r_0\}$$

We will build the extension of $\tilde{\Theta}$ into $\mathcal{O}^c$ (which is the inner cylinder) by symmetrization with respect to the cylinder S in a small neighborhood of S. We define the symmetric function as follows:

$$\begin{cases} \tilde{\Theta}^\#(x, r, \alpha) = \tilde{\Theta}(x, r, \alpha), & r_0 < r < R_0 \\ \tilde{\Theta}^\#(x, r, \alpha) = \tilde{\Theta}(x, 2r_0 - r, \alpha)\rho_h(r_0 - r), & r < r_0 \end{cases} \tag{2.6}$$

where $0 < h < r_0/2$, and $0 \le \rho_h(s) \le 1$ is any smooth cut-off function such that

$$\begin{cases} \rho_h(s) = 0, & s > h \\ \rho_h(s) = 1, & s < h/2 \end{cases}$$

Note that the symmetric function $\tilde{\Theta}^\#$ is zero on the two lateral discs spanned by Γ_{in} and Γ_{out}. This is due to the fact that $\tilde{\Theta} \in H^1_{\Gamma'}$. Moreover, by construction, we have that $\tilde{\Theta} \in H^1_0(D)$.

Finally, we bring the symmetric function $\tilde{\Theta}^\#$ back on the physical domains Ω and Ω^c. Let $\bar{u} \in C^2(D)$ be any extension of $u \in C^2(\mathcal{O})$ to the whole domain D. For example, we can take

$$\bar{u}(x, r, \alpha) = u \circ p_{\mathcal{O}} \cdot \rho_h(r_0 - r), \tag{2.7}$$

where $p_{\mathcal{O}} = I - b_{\mathcal{O}} \nabla b_{\mathcal{O}}$ is the usual projection on S.

Now we define $\Psi = \tilde{\Theta}^\# \circ (I + \bar{u})^{-1}$, which is our specific extension of Θ in $H^1_0(D)$ (due to the regularity and smoothness and $\tilde{\Theta}^\#$ and $(I + \bar{u})^{-1}$, respectively). $\square$

Corollary 2.1. *Let $\Theta \in H^1_{\Gamma'}(\Omega)$. If θ is its trace on Γ, then $\theta \in H^{1/2}_{00}(\Gamma)$.*

The corollary follows immediately from Lemma 2.2.

Bilinear Forms: We define the following bilinear forms: $\forall (U', \Theta) \in H^1_{\Gamma'}(\Omega)^2$,

$$\begin{aligned} A^0_{u,p}(U', \Theta) &= \int_{\Omega_u} \mathcal{T}'(U')..D\Theta \, dx + \int_{\Gamma_u} \langle (\mathcal{T} - pI)(D^*U')n, \, \Theta \rangle \, d\Gamma_u \\ &= \int_{\Omega_u} \langle \mathrm{Div}(\mathcal{T}'(U')), \Theta \rangle \, dx \\ &\quad + \int_{\Gamma_u} \{ \langle \mathcal{T}'(U')n, \, \Theta \rangle + \langle (\mathcal{T} - pI)(D^*U')n, \, \Theta \rangle \} \, d\Gamma \end{aligned} \tag{2.8}$$

and

$$\begin{aligned} A^1_{u,p}(U', \Theta) &= A^0_{u,p}(U', \Theta) \\ &\quad - \int_{\Gamma_u} \big\langle \nabla p \cdot n \; (U' \cdot n)n + (pI - \mathcal{T})(D^2 b_\Omega)U' - U' \cdot n \; \mathrm{Div}_\Gamma(\mathcal{T}), \, \Theta \big\rangle \, d\Gamma. \end{aligned} \tag{2.9}$$

Then we see that U' solves the following variational problem:

$$U' \in H^1_{\Gamma'}(\Omega_u), \; s.t., \; \forall \Theta \; in \; H^1_{\Gamma'}(\Omega_u), \; A^1_{u,p}(U', \Theta) = \int_\Gamma \langle G, \Theta \rangle \, d\Gamma \tag{2.10}$$

where $G = (p'I - 2\epsilon(w'))n$ on Γ.

Using Lemma 2.1, Corollary 2.1, and (2.3), we can see that (2.10) is equivalent to: $U' \in H^1_{\Gamma'}(\Omega_u)$, $s.t.$, $\forall \Theta$ in $H^1_{\Gamma'}(\Omega_u)$,

$$A^1_{u,p}(U', \Theta) = \int_{\Omega^c} [\, 2\varepsilon(w')..\varepsilon(R_\Gamma(\gamma_\Gamma \Theta)) - p' \, \mathrm{div}(R_\Gamma(\gamma_\Gamma \Theta)) \,] dx, \tag{2.11}$$

where $\gamma_\Gamma \in \mathcal{L}(H^1_{\Gamma'}(\Omega), H^{1/2}_{00}(\Gamma))$ is the trace operator on Γ.

Note that the bilinear forms introduced above are defined on domains and boundaries which move with u. Thus, our next goal is to transport them back on the fixed geometry $\mathcal{O}, S$.

Step 2: Using (2.8) and (2.9), we build non-symmetrical bilinear forms on $H^1_{\Gamma'}(\mathcal{O})$.

Our objective is to show that when both p and u are sufficiently small, the bilinear form $A^1_{u,p}$ is coercive. The idea is to use a perturbation of Lax-Milgram setting for the coercive bilinear form $A^1_{0,0}$. Note that the Banach spaces associated to this family of bilinear forms depend on u. For a correct perturbation setting we need to "bring back" these bilinear forms to the fixed space $\mathcal{O}$. Thus, for $i = 0, 1$, we consider

$$a^i_{u,p}(u', \theta) := A^i_{u,p}(u' \circ (I + u)^{-1}, \Theta), \;\; with \;\; \Theta = \theta \circ (I + u)^{-1}.$$

Notice that when Θ ranges in the space $H^1_{\Gamma'}(\Omega_u)$, θ ranges in the space $H^1_{\Gamma'}(\mathcal{O})$.

First we transport the bilinear form $A^0_{u,p}$ to find the explicit formula for $a^0_{u,p}$. Using the change of variable $\varphi = I + u$, (i.e., $\Omega_u = \varphi(\mathcal{O})$), and the usual notation

$J = \det(D\varphi) = \det(I + Du)$, we obtain

$$a^0_{u,p}(u', \theta) = \int_{\mathcal{O}} \mathcal{T}'(U') \circ \varphi..(D\Theta) \circ \varphi \, J \, dx \; + \int_{\mathcal{O}} \mathrm{div}[D_{\Gamma_u} U'(\mathcal{T} - pI)\Theta\,] \circ \varphi \, J \, dx$$

$$= \int_{\mathcal{O}} \mathcal{T}'(U') \circ \varphi..(D\theta)(D\varphi)^{-1} \, J \, dx + \int_{\mathcal{O}} \mathrm{div}\{J\,(D\varphi)^{-1}[(D_{\Gamma_u} U'(\mathcal{T} - pI)) \circ \varphi]\,\theta\,\}\, dx$$

$$= \int_{\mathcal{O}} \mathcal{T}'(U') \circ \varphi..D\theta(D\varphi)^{-1} \, J \, dx \; + \int_S \{J\,(D\varphi)^{-1}[(D_{\Gamma_u} U'(\mathcal{T} - pI)) \circ \varphi]\,\theta\,\} \cdot n_S dS \tag{2.12}$$

In the above expression for $a^0_{u,p}$, n_S stands for the unit outer normal vector along the boundary S with respect to $\mathcal{O}$.

We begin with the first term on the right side of (2.12). Using expression (1.11) for $\mathcal{T}'$, we obtain

$$\begin{aligned}\mathcal{T}'(U') \circ (I + u) \; = & - \frac{div(u')}{det(I + Du)} \; (I + Du)[C_{\lambda,\mu}..(\sigma(u))](I + D^*u) \\ & + \Big(\frac{1}{det(I + Du)} D(u')[C_{\lambda,\mu}..(\sigma(u))](I + D^*u)\Big) \\ & + \Big(\frac{1}{det(I + Du)} (I + Du)[C_{\lambda,\mu}..(\sigma')](I + D^*u)\Big) \\ & + \Big(\frac{1}{det(I + Du)} (I + Du)[C_{\lambda,\mu}..(\sigma(u))]D^*(u')\Big) \\ & - \; [\mathbf{D}\mathcal{T} \circ (I + u)]u' \end{aligned} \tag{2.13}$$

At this point, for simplicity and the sake of exposition, we assume that the elastic material is incompressible so that $\mathrm{div}(u) = \mathrm{div}(u') = 0$. Also, we introduce the following notation. Let

$$\begin{aligned}\epsilon_u(u') = \quad & D(u')(C_{\lambda,\mu}..(\sigma(u))) + (I + Du)C_{\lambda,\mu}..(\sigma') \\ & + (I + Du)[C_{\lambda,\mu}..(\sigma(u))]D^*(u')(I + D^*u)^{-1}\end{aligned} \tag{2.14}$$

Combining (2.13) with (2.14), we can rewrite

$$\mathcal{T}'(U') \circ (I + u) = \frac{1}{J} \; \epsilon_u(u')(I + D^*u) - [\mathbf{D}\mathcal{T} \circ (I + u)]u' \tag{2.15}$$

Next we deal with the second term on the right side of (2.12). We have that

$$[D_{\Gamma_u} U'] \circ (I + u) = D_S u'(D\varphi)^{-1}\,[I - B] \; + \; Du'.N \tag{2.16}$$

where

$$N = n_S \otimes n_S (D\varphi)^{-1}\,[I - B]$$

and

$$B = ||(D\varphi)^{-*} n_S||^{-2} \; (D\varphi)^{-*} n_s \otimes (D\varphi)^{-*} n_S$$

Using (2.15) and (2.16) back into (2.12), we obtain that

$$a^0_{u,p}(u', \theta) = \int_{\mathcal{O}} (1/J\,\epsilon_u(u')(I + D^*u) - (\mathcal{D}\mathcal{T} \circ \varphi)u')..(D\theta(D\varphi)^{-1}\,)\,J\,dx$$

$$+ \int_S \{J\,(D\varphi)^{-1}(D_S u'(D\varphi)^{-1}\,[I - B] \, + \, Du'N)[(\mathcal{T} - pI) \circ (I + u)]\theta\,\} \cdot n_S dS. \tag{2.17}$$

To further simplify the expression of $a^0_{u,p}$, we use the "tangential" property of the term $\mathcal{T} - pI$. For any $x \in S$, the vector $[(\mathcal{T} - pI) \circ (I + u)(x)]\theta(x)$ is tangent to Γ_u at the point $X = x + u(x) \in \Gamma_u$, which means that

$$\forall\theta \in H^1(\mathcal{O}),\ \forall x \in S,\ \ [(\mathcal{T} - pI) \circ (I + u)(x)]\theta(x) \in T_X\Gamma_u,\ \ \textit{with}\ \ X = x + u(x)$$

Using the fact that

$$\begin{aligned}&\big\langle (D\varphi)^{-*}(x) n_S(x), [(\mathcal{T} - pI) \circ (I + u)(x)]\theta(x)\big\rangle = \\ &= \big\langle n_S(x), (D\varphi)^{-1}(x)[(\mathcal{T} - pI) \circ (I + u)(x)]\theta(x)\big\rangle = 0\end{aligned} \tag{2.18}$$

and since $(D\varphi)^{-1}(x) = D(\varphi^{-1}) \circ \varphi$ applies $T_X\Gamma_u$ on to T_xS, we obtain that

$$B[(\mathcal{T} - pI) \circ (I + u)(x)]\theta(x) = 0 \tag{2.19}$$

By the same argument, we have that :

$$N(x)[(\mathcal{T} - pI) \circ (I + u)(x)]\theta(x) = 0 \tag{2.20}$$

Combining (2.17) with (2.19) and (2.20), we obtain our final expression for $a^0_{u,p}$:

$$\begin{aligned}a^0_{u,p}(u',\theta) = &\int_{\mathcal{O}} \epsilon_u(u')(D^*\varphi)..D\theta(D\varphi)^{-1}dx - \int_{\mathcal{O}} [\mathcal{D}\mathcal{T} \circ \varphi]u'..D\theta(D\varphi)^{-1}\ J\, dx \\ &+ \int_S \{J\,(D\varphi)^{-1}\,D_S u'\,(D\varphi)^{-1}\ [(\mathcal{T} - pI) \circ \varphi]\theta\,\} \cdot n_S dS.\end{aligned}$$

Finally, using the expression for $a^0_{u,p}$ and the formula for change of variable on the boundary, we obtain the transported bilinear form $a^1_{u,p}$:

$$\begin{aligned}a^1_{u,p}(u',\theta) = a^0_{u,p}(u',\theta) - &\int_S \Big\langle (\nabla p \circ \varphi) \cdot (n \circ \varphi)\ u' \cdot (n \circ \varphi)(n \circ \varphi) \\ &+ ((p \circ \varphi)I - \mathcal{T} \circ \varphi)(D^2 b_{\Omega_u} \circ \varphi)u' - u' \cdot (n \circ \varphi)\,(\mathrm{Div}_\Gamma(\mathcal{T}) \circ \varphi),\,\theta\Big\rangle\ \omega(u)\ dS\end{aligned}$$

with $\omega = \det(I + Du)||(I + Du)^{-*}n_S||$ and $n = n_{\Gamma_u}$.

Step 3: Equicontinuity for the bilinear form $a^1_{u,p}(u', u')$.

Recall the definition of the set B_ρ:

$$B_\rho = \{(u,p) \in C^2(\bar{\mathcal{O}}) \times C^1(\bar{\Omega}^c)\ |\ ||u||_{C^2(\bar{\mathcal{O}})} \le \rho,\ ||q||_{C^1(\bar{\mathcal{O}}^c)} \le \rho\}$$

where $q = p \circ (I + \bar{u})$, with $\bar{u} \in C^2(D)$ being any extension of $u \in C^2(\mathcal{O})$ to the whole domain D.

Now we define the following set:

$$S^1 = \{\, f \in H^1_{\Gamma'}(\mathcal{O}),\ s.t.\ ||f||_{H^1_{\Gamma'}(\mathcal{O})} = 1\,\}$$

Proposition 2.1. *The mapping* $(u,p) \to a^1_{u,p}(u', u')$ *is equicontinuous from* B_ρ *into* $\mathbb{R}$ *with respect to* $u' \in S^1$.

Proof. The proof is done in several steps. Since $||u||_{C^2(\bar{\mathcal{O}})} \le \rho$, then there exists $M = M(\rho)$ such that

$$\forall u \in B_\rho,\quad ||I + u||_{C^2(\bar{\mathcal{O}})} \le M,\ \ ||(I + u)^{-1}||_{C^2(\bar{\mathcal{O}})} \le M$$

We also have the following 'controlled norm' on the trace operator $\gamma_{\Gamma_u} \in \mathcal{L}(H^1(\Omega_u), H^{1/2}(\Gamma_u))$:

$$\exists K > 0\ \ s.\,t.\ \ \forall u \in B_\rho,\quad ||\gamma_{\Gamma_u}||_{\mathcal{L}(H^1(\Omega_u),H^{1/2}(\Gamma_u))} \le K.$$

Since $u' \in S^1$, we have the following estimates:

$$||u'||_{H^{1/2}(S)} \leq K$$

and

$$||D_S u'||_{H^{-1/2}(S)} \leq cK.$$

Regarding $U' \in H^1(\Omega_u)$, we have similar uniform bounds:

$$||U'||_{H^1(\Omega_u)} = ||u' \circ \varphi^{-1}||_{H^1(\Omega_u)} \leq ||u'||_{H^1(\mathcal{O})} \cdot ||\varphi^{-1}||_{W^{1,\infty}(\mathcal{O})} \leq M.$$

and

$$||U'||_{H^{1/2}(\Gamma_u)} \leq ||\gamma_{\Gamma_u}|| \cdot ||U'||_{H^1(\Omega_u)} \leq K\,M.$$

Therefore, using again the continuity of the tangential derivative on Γ_u, we obtain:

$$||D_{\Gamma_u}(U')||_{H^{-1/2}(\Gamma_u)} \leq c||\gamma_{\Gamma_u}|| \cdot ||U'||_{H^{1/2}(\Gamma_u)} \leq c\,K\,M.$$

Note that the integrands that appear in the expression of $a^1_{u,p}$ are polynomial products of two types of terms. The first type consists of terms depending on u and p, which are continuous with respect to u and p as functions defined from B_ρ into $C^0(\bar{\mathcal{O}})$. The other type is made of the terms depending on u', which are bounded in L^2. Thus taking advantage of all the uniform bounds obtained above we obtain the continuity of the mapping

$$(u,p) \to a^1_{u,p}(u',u'),$$

this continuity being uniform with respect to u' in the unit sphere S^1. □

Step 4: Coercivity for the bilinear form $a^1_{u,p}(u',u')$.

For $p=0$, and implicitly $u=0$ and $\mathcal{T}=0$, we have that $a^1_{0,0}$ is coercive, so that there exists $\alpha_0 > 0$ such that $a^1_{0,0}(u',u') \geq \alpha_0\,||u'||^2$.

Now we consider

$$\pi_{u,p}(u') := \frac{a^1_{u,p}(u',u')}{||u'||^2_{H^1_{\Gamma'}(\mathcal{O})}}$$

As the mapping $u \to a^1_{u,p}(u',u')$ is equicontinuous in B_ρ with respect to u' ranging in $S^1 = \{u' \in H^1_{\Gamma'}(\mathcal{O})$, s.t. $||u'||^2_{H^1_{\Gamma'}(\mathcal{O})} = 1\}$, then the mapping $(u,p) \to \pi_{u,p}(u')$ is equicontinuous in B_ρ with respect to u' ranging in $H^1_{\Gamma'}(\mathcal{O})$. Then $\inf_{u' \in S^1} \pi_{u,p}(u')$ is continuous on B_ρ. Thus there exists $\rho^* > 0$ small such that if $(u,p) \in B_\rho$, for any $\rho \leq \rho^*$, we have that $\inf_{u' \in S^1} \pi_{u,p}(u') \geq \alpha_0/2$. This is equivalent to $\frac{\alpha_0}{2}$ - coercivity of $a^1_{u,p}(u',u')$.

Thus for ρ^* sufficiently small, the bilinear form $a^1_u(u',u')$ is coercive on the Sobolev space $H^1_{\Gamma'}(\mathcal{O})$. Then by the Lax-Milgram theorem for non-symmetrical coercive forms, we obtain the existence of a unique solution u' to the linearized problem (2.4) transported back on the fixed reference geometry $\mathcal{O}, S$. Readily, the transported element $U' := u' \circ (I+u)^{-1}$ is the unique solution in $H^1_{\Gamma'}(\Omega_u)$ to the problem (2.4). □

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