

## HARMONIC LIMITS OF DYNAMICAL SYSTEMS

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**ABSTRACT.** In this paper, we analyze the rotational behaviour of dynamical systems, particularly of solutions of ODEs. With rotational behaviour we mean the existence of rotational factor maps, i.e., semi-conjugations to rotations in the complex plane. In order to analyze this kind of rotational behaviour, we introduce harmonic limits  $\lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T e^{i\omega t} f(\Phi_t x) dt$ . We discuss the connection between harmonic limits and rotational factor maps, and some properties of the limits, e.g., existence under the presence of an invariant measure by the Wiener-Wintner Ergodic Theorem. Finally, we look at linear differential equations (autonomous and periodic), and show the connection between the frequencies of the rotational factor maps and the imaginary parts of the eigenvalues of the system matrix (or of the Floquet exponents in the periodic case).

**1. Introduction.** In a linear differential equation  $\dot{x} = Ax$  in  $\mathbb{C}^n$ ,  $A \in \mathbb{C}^{n \times n}$ , the real parts of the eigenvalues of the matrix  $A$  describe how fast trajectories grow or decay. These real parts can be generalized for nonlinear systems to Lyapunov exponents, which describe how fast two trajectories starting close to each other separate over time. On the other hand, the imaginary parts of the eigenvalues of  $A$  describe the rotational behaviour of the solutions of the linear differential equation. There have been several attempts to generalize them to nonlinear systems, and to describe the rotational behaviour of dynamical systems in general. See, e.g., [1], [5], [6], or [7] for discussions of rotation numbers in continuous time.

In this paper, we will analyze a different approach to describe the rotational behaviour, namely the concept of rotational factor maps, which was introduced for discrete-time systems in [3, Section 3.1] and further pursued, e.g., in [2] and [4]. The general idea is to find a complex-valued map  $F$  on the state space that maps the dynamics onto a rotation around the origin in the complex plane. More formally, for a semi-flow  $\Phi_t$  on a metric space  $X$ , we will look for a map  $F : X \rightarrow \mathbb{C}$ ,  $F \neq 0$ , with  $F \circ \Phi_t = e^{i\omega t} \cdot F$  for some  $\omega \in \mathbb{R}$  and all  $t \geq 0$ . We will call such a map a *rotational factor map* to the frequency  $\omega/2\pi$ .

This concept of rotational factor maps is closely connected to harmonic limits, which are defined for  $f : X \rightarrow \mathbb{C}$ ,  $\omega \in \mathbb{R}$  and  $x \in X$  by

$$f_\omega^*(x) := \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T e^{i\omega t} f(\Phi_t x) dt,$$

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if the limit exists. It turns out that there is a rotational factor map to some frequency  $\omega/2\pi$  if and only if there is a map  $f : X \rightarrow \mathbb{C}$  such that the harmonic limit  $f_\omega^*$  is not constant zero. Because of this equivalence, harmonic limits will be the key object of our analysis.

In [3], Mezić and Banaszuk use harmonic limits as a tool to compare dynamical systems, and in order to determine model parameters. We are interested in the properties of the harmonic limits themselves instead. We will, e.g., investigate existence of the harmonic limit under the presence of an invariant measure, and show how periodicity properties of the system affect the frequencies that can occur.

**2. Rotational factor maps and harmonic limits.** Let a metric space  $X$  and a semi-flow  $\Phi_t : X \rightarrow X$  be given. We want to analyze the rotational behaviour of  $\Phi_t$  in the sense of the existence of rotational factor maps.

**Definition 2.1** (Rotational factor maps). A map  $F : X \rightarrow \mathbb{C}$  that satisfies  $F \not\equiv 0$  and

$$F \circ \Phi_t = e^{it\omega} \cdot F \tag{1}$$

for all times  $t \geq 0$  and a number  $\omega \in \mathbb{R}$  is called a *rotational factor map* with frequency  $\omega/2\pi$ .

This means, we are looking for semi-conjugations to rotations around the origin of the complex plane. Note that the condition  $F \not\equiv 0$  in this definition is necessary, because, with  $F \equiv 0$ , the semi-conjugacy (1) trivially holds for all systems  $\Phi_t$  and all  $\omega \in \mathbb{R}$ . Further note that, for  $F \equiv c \neq 0$  and  $\omega = 0$ , the semi-conjugacy (1) also holds for all systems  $\Phi_t$ , which implies that every system has a rotational factor map with frequency 0.

Rotational factor maps can also be interpreted as eigenfunctions of the semi-group of Koopman operators given by  $U_t f := f \circ \Phi_t$ ,  $t \geq 0$ .

Consider the following simple example.

**Example 1.** Let  $X := \mathbb{R}^2$  and  $\alpha \in \mathbb{R}$ , and consider the system given by the linear differential equation

$$\dot{x} = \begin{pmatrix} 0 & -\alpha \\ \alpha & 0 \end{pmatrix} x,$$

i. e., consider

$$\Phi_t x := \begin{pmatrix} \cos \alpha t & -\sin \alpha t \\ \sin \alpha t & \cos \alpha t \end{pmatrix} x.$$

This is a rotation around the origin with frequency  $\alpha/2\pi$ . So all we have to do in order to construct a rotational factor map with frequency  $\alpha/2\pi$  is to identify  $\mathbb{R}^2$  with  $\mathbb{C}$ . The map  $F : \mathbb{R}^2 \rightarrow \mathbb{C}$  given by  $F(x_1, x_2) := x_1 + ix_2$  indeed is a rotational factor map with frequency  $\alpha/2\pi$ , because

$$\begin{aligned} F(\Phi_t x) &= F(x_1 \cos \alpha t - x_2 \sin \alpha t, x_1 \sin \alpha t + x_2 \cos \alpha t) \\ &= x_1 \cos \alpha t - x_2 \sin \alpha t + ix_1 \sin \alpha t + ix_2 \cos \alpha t \\ &= x_1 e^{i\alpha t} + ix_2 e^{i\alpha t} \\ &= e^{i\alpha t} (x_1 + ix_2) \\ &= e^{i\alpha t} F(x). \end{aligned}$$

In general, existence of rotational factor maps does not imply that the system rotates in a geometrical sense, especially if the factor map is zero almost everywhere with respect to an invariant measure. Consider this example:

**Example 2.** Let  $X := [0, 1]$  and  $\dot{x} = -x$ , i. e.,  $\Phi_t x = e^{-t}x$ . Choose  $\omega \in \mathbb{R}$ , and let

$$F : X \rightarrow \mathbb{C}, \quad x \mapsto \begin{cases} e^{-i\omega \log x} & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

This map satisfies (1) for any  $\omega$ , so there is a rotational factor map to any frequency, although the system does not rotate in a geometrical sense.

Usually, it is not as easy to find a rotational factor map as in Example 1. The dynamics might be much more complicated, or even not exactly known. So the question is, if it is possible to prove the existence of rotational factor maps in those cases. In fact, this is possible with the help of harmonic limits.

**Definition 2.2** (Harmonic limit). For  $f : X \rightarrow \mathbb{C}$ ,  $\omega \in \mathbb{R}$  and  $x \in X$ , the *harmonic limit* is given by

$$f_\omega^*(x) := \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T e^{it\omega} f(\Phi_t x) dt,$$

if the limit exists.

**Remark 1.** The harmonic limit can also be defined in the same way not only for semi-flows but, more generally, for any function  $\Phi : \mathbb{R}^+ \times X \rightarrow X$ ,  $(t, x) \mapsto \Phi_t x$ , that is Borel measurable in  $t$  and satisfies  $\Phi_0 x = x$ .

Note that, even if one only has access to a complex-valued output  $f$  of the system, one can still compute this harmonic limit. Harmonic limits can be used as a tool to analyze the rotational behaviour, because they indicate the existence of rotational factor maps, as will be shown in the following theorem.

In the following, we will always assume that  $f(\Phi_t x)$  is locally integrable in  $t$ , wherever this term appears.

**Theorem 2.3.** *Let  $\omega \in \mathbb{R}$ . There is a rotational factor map  $F$  with frequency  $\omega/2\pi$  if and only if there is a map  $f : X \rightarrow \mathbb{C}$  such that  $f_\omega^* \neq 0$ .*

Note that in this theorem,  $f_\omega^*$  exists at all points  $x \in X$ . If there is an invariant measure  $\mu$  on  $X$ , one can also consider maps  $F$  that are rotational factor maps  $\mu$ -almost everywhere, i. e., maps, which satisfy (1) only almost everywhere, see [8, Remark 2.2.3]. For these, Theorem 2.3 also holds, but then  $f_\omega^*$  also exists only almost everywhere.

The proof of this theorem is actually almost trivial. In fact, if  $f_\omega^* \neq 0$ , then the complex conjugate of this harmonic limit is a rotational factor map.

*Proof of Theorem 2.3.* Let  $f : X \rightarrow \mathbb{C}$  and  $\omega \in \mathbb{R}$  be such that  $f_\omega^* \neq 0$ . Let  $F := \overline{f_\omega^*}$ . Then

$$\begin{aligned} \overline{F(\Phi_\tau x)} &= f_\omega^*(\Phi_\tau x) = \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T e^{it\omega} f(\Phi_{t+\tau} x) dt = \lim_{T \rightarrow \infty} \frac{1}{T} \int_\tau^{T+\tau} e^{i(t-\tau)\omega} f(\Phi_t x) dt \\ &= e^{-i\tau\omega} \lim_{T \rightarrow \infty} \frac{1}{T} \left[ \int_0^{T+\tau} e^{it\omega} f(\Phi_t x) dt - \int_0^\tau e^{it\omega} f(\Phi_t x) dt \right] = e^{-i\tau\omega} f_\omega^*(x) \end{aligned}$$

for every  $\tau \geq 0$  and all  $x \in X$ . So  $F \circ \Phi_\tau = e^{i\tau\omega} \overline{f_\omega^*} = e^{i\tau\omega} F$ , i. e.,  $F$  is a rotational factor map to the frequency  $\omega/2\pi$ .

On the other hand, let  $F : X \rightarrow \mathbb{C}$  be a rotational factor map to the frequency  $\omega/2\pi$ ,  $\omega \in \mathbb{R}$ , and let  $f := \overline{F}$ . Then

$$\begin{aligned} f_\omega^*(x) &= \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T e^{it\omega} f(\Phi_t x) dt = \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T e^{it\omega} \overline{F(\Phi_t x)} dt \\ &= \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T e^{it\omega} \overline{e^{it\omega} F(x)} dt = \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T \overline{F(x)} dt = \overline{F(x)} \end{aligned}$$

for all  $x \in X$ . So  $f_\omega^* = \overline{F}$ , which, by definition of rotational factor maps, is not constant zero. □

By this theorem, in order to prove existence of a rotational factor map, it suffices to show that there is a non-vanishing harmonic limit. Furthermore, all rotational factor maps can be detected in this manner.

**3. Properties of harmonic limits.** Because of the equivalence between non-vanishing harmonic limits and the existence of rotational factor maps proposed in Theorem 2.3, we will discuss some properties of rotational factor maps in this section.

**Remark 2.** With the exception of Theorem 3.1, all results in this section not only hold for semi-flows but also for more general functions  $\Phi$  as in Remark 1.

**3.1. Existence.** The limit in the definition of  $f_\omega^*$  does not necessarily exist. So we give a criterion for existence here.

**Theorem 3.1** (Wiener-Wintner). *Assume that there is a finite invariant measure  $\mu$  on  $X$  such that the map  $f : X \rightarrow \mathbb{C}$  is of class  $L^p(\mu)$ ,  $1 \leq p < \infty$ . Then there is a null set  $\Xi \subset X$ , such that the harmonic limit  $f_\omega^*(x)$  exists for all  $x \in X \setminus \Xi$  and every  $\omega \in \mathbb{R}$ . Furthermore, the map  $x \mapsto f_\omega^*(x)$  is of class  $L^p(\mu)$ .*

*Proof.* This can be shown with the Wiener-Wintner Ergodic Theorem in combination with an ergodic partitioning, see [8, Theorem 2.3.18]. □

By this theorem, the harmonic limit exists almost everywhere, independently of the frequency.

**3.2. Periodicity.** At periodic and quasi-periodic points, the periods determine the frequencies that can occur. Let us first look at periodic points. To calculate the harmonic limit at periodic points, it suffices to compute a finite integral.

**Proposition 1.** *If  $x \in X$  is such that  $\Phi_\tau x = x$  for some  $\tau > 0$ , then  $f_{2k\pi/\tau}^*(x) = 1/\tau \cdot \int_0^\tau e^{it2k\pi/\tau} f(\Phi_t x) dt$  for every  $f : X \rightarrow \mathbb{C}$  and all  $k \in \mathbb{Z}$ . Particularly,  $f_{2k\pi/\tau}^*(x)$  exists.*

*Proof.* See [8, Proposition 2.3.28]. □

The following proposition characterizes the frequencies that can possibly occur at periodic points.

**Proposition 2.** *Let  $x \in X$ ,  $f : X \rightarrow \mathbb{C}$ , and  $\tau > 0$  be such that  $f(\Phi_t x)$  is  $\tau$ -periodic in  $t$ . If  $\omega \in \mathbb{R}$  is such that  $\omega\tau/2\pi \notin \mathbb{Z}$ , then  $f_\omega^*(x) = 0$ .*

*Proof.* See [8, Proposition 2.3.33]. □

By this proposition, only frequencies  $\omega/2\pi$  that are integer multiples of the frequency  $1/\tau$  can occur. But for a given map  $f$ , the harmonic limit to such a frequency might be zero, though. The following theorem shows that, under a weak condition, all those frequencies really occur and can be detected with harmonic limits of continuous functions.

**Theorem 3.2.** *Let  $\omega > 0$ . Let  $X$  be a metric space and  $p : \mathbb{R} \rightarrow X$  a continuous  $2\pi/\omega$ -periodic function. If there is a nontrivial open interval  $J \subset [0, 2\pi/\omega)$  such that  $p$  is injective on  $J$ , then for every  $k \in \mathbb{Z}$ , there is  $f : X \rightarrow \mathbb{C}$  continuous, such that  $\int_0^{2\pi/\omega} e^{ik\omega t} f(p(t)) dt \neq 0$ .*

*In particular, if  $\Phi_t x = p(t)$  for some  $x \in X$ , then for every  $k \in \mathbb{Z}$ , there is  $f : X \rightarrow \mathbb{C}$  continuous, such that  $f_{k\omega}^*(x) \neq 0$ .*

*Proof.* See [8, Theorem 2.3.38]. □

The condition in this theorem, that  $p$  is injective on a nontrivial open interval, for example is satisfied if  $p$  is continuously differentiable, see the following corollary. So in particular, Theorem 3.2 is applicable, if  $\Phi_t x$  is the solution of a differential solution with continuous right-hand side.

**Corollary 1.** *Let  $X$  be an  $n$ -dimensional normed linear space over  $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ ,  $n \in \mathbb{N}$ . Let  $p : \mathbb{R} \rightarrow X$  be  $2\pi/\omega$ -periodic,  $\omega > 0$ . If there is a nontrivial open interval  $I \subset [0, 2\pi/\omega)$  such that  $p|_I$  is continuously  $\mathbb{K}$ -differentiable and not constant, then for every  $k \in \mathbb{Z}$ , there is  $f : X \rightarrow \mathbb{C}$  continuous, such that  $\int_0^{2\pi/\omega} e^{ik\omega t} f(p(t)) dt \neq 0$ .*

*Proof.* See [8, Corollary 2.3.40]. □

Also at quasi-periodic orbits one can characterize the set of possible frequencies. In this context, we define quasi-periodic maps as follows.

**Definition 3.3** (Quasi-periodicity). A map  $q : \mathbb{R} \rightarrow \mathbb{C}$  is called *quasi-periodic* with periods  $\tau_j$ ,  $j = 1, \dots, n$ , if there is a locally Lebesgue-integrable map  $Q : \mathbb{R}^n \rightarrow \mathbb{C}$  that is  $\tau_j$ -periodic in its  $j$ -th argument,  $j = 1, \dots, n$ , such that  $Q(t, \dots, t) = q(t)$  for all  $t \in \mathbb{R}$ .

**Proposition 3.** *Let  $x \in X$  and  $f : X \rightarrow \mathbb{C}$  be such that  $f(\Phi_t x)$  is quasi-periodic in  $t$  with periods  $\tau_j$ ,  $j = 1, \dots, n$ . If  $\omega \in \mathbb{R}$  is such that the numbers  $\omega, 2\pi/\tau_1, \dots, 2\pi/\tau_n$  are rationally independent, then  $f_\omega^*(x) = 0$ .*

*Proof.* See [8, Proposition 2.3.32]. □

For almost periodic orbits, one can show existence of the harmonic limit without referring to invariant measures as in Theorem 3.1.

**Proposition 4.**

1. *If  $f : X \rightarrow \mathbb{C}$  is continuous, and  $x \in X$  is such that  $\Phi_t x$  is almost periodic in  $t$ , then  $f_\omega^*(x)$  exists for all  $\omega \in \mathbb{R}$ .*
2. *If  $x \in X$  and  $f : X \rightarrow \mathbb{C}$  are such that  $f(\Phi_t x)$  is almost periodic in  $t$ , then  $f_\omega^*(x)$  exists for all  $\omega \in \mathbb{R}$ .*

*Proof.* See [8, Proposition 2.3.29]. □

**3.3. Asymptotics.** Not only periodicity impacts the properties of harmonic limits, also the asymptotic behaviour of the system can be used in the analysis. If, e. g., two trajectories asymptotically approach each other, the harmonic limits to a continuous function  $f$  coincide on both trajectories. More generally, the following holds.

**Proposition 5.** *Let  $X$  be a metric space,  $x \in X$  and  $T_0 \geq 0$ . Let  $g : [T_0, \infty) \rightarrow \mathbb{C}$  be locally integrable. If  $g(t) - f(\Phi_t x) \rightarrow 0$  for  $t \rightarrow \infty$ , then it holds for every  $\omega \in \mathbb{R}$ , that*

$$f_\omega^*(x) = \lim_{T \rightarrow \infty} \frac{1}{T} \int_{T_0}^T e^{i\omega t} g(t) dt,$$

provided that one of the limits exists.

*Proof.* See [8, Proposition 2.3.26]. □

**4. Harmonic limits of linear differential equations.** We will now apply these concepts and results to solutions of ordinary linear differential equations. We will see that the occurring frequencies are determined by the eigenvalue imaginary parts of the system matrix in the autonomous case (Section 4.1), and by the imaginary parts of the Floquet exponents in the periodic case (Section 4.2).

In both cases, we will project the solutions onto the unit sphere, and analyze the resulting flow instead. This has the advantage that we will deal with a compact state space. The rotational behaviour of the projected flow in the sense of rotational factor maps is essentially the same as the behaviour of the original system.

**4.1. Autonomous equations.** Consider the linear differential equation  $\dot{x} = Ax$  in  $\mathbb{C}^n$  for  $A \in \mathbb{C}^{n \times n}$ . We project its solution onto the unit sphere  $S^{n-1} \subset \mathbb{C}^n$ . This yields the flow

$$\Phi_{ts} := \frac{e^{At}s}{\|e^{At}s\|}$$

on  $S^{n-1}$ , where  $\|\cdot\|$  denotes the Euclidean norm.

For every starting point  $s \in S^{n-1}$ , the flow  $\Phi_{ts}$  asymptotically approaches a quasi-periodic orbit  $Q_{ts}$  with periods given by  $2\pi/\Im\lambda$ ,  $\lambda \in \text{spec}_s A$ , where  $\text{spec}_s A$  is a subset of  $\text{spec} A$  determined by the point  $s$ . More precisely,  $\text{spec}_s A$  is given as follows.

**Definition 4.1** ( $\text{spec}_s A$ ). Let  $A \in \mathbb{C}^{n \times n}$  and  $s \in \mathbb{C}^n$ . Let  $P \in \mathbb{C}^{n \times n}$  be such that  $P^{-1}AP$  is in Jordan normal form with the eigenvalues  $\lambda_1, \dots, \lambda_n \in \mathbb{C}$  on its diagonal, and define  $J := \{j \in \{1, \dots, n\} \mid s_j \neq 0\}$ . Let  $\Re\lambda_{\max} := \{\max \Re\lambda_j \mid j \in J\}$ . Then we define  $\text{spec}_s A := \{\lambda \in \text{spec} A \mid \Re\lambda = \Re\lambda_{\max}, \lambda = \lambda_j \text{ for some } j \in J\}$ .

Note that  $Q_{ts}$  is continuous in  $t$ . For more details, see [8, Section 2.5].

**Proposition 6.** *Let  $A \in \mathbb{C}^{n \times n}$ . Then for every  $s \in S^{n-1}$ , every continuous map  $f : S^{n-1} \rightarrow \mathbb{C}$ , and all  $\omega \in \mathbb{R}$ , the harmonic limit  $f_\omega^*(s)$  exists, and it holds that*

$$f_\omega^*(s) = \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T e^{i\omega t} f(Q_{ts}) dt. \tag{2}$$

*Proof.* By [8, Lemma 2.5.13], it holds that  $\Phi_{ts} - Q_{ts} \rightarrow 0$  for  $t \rightarrow \infty$ . So Equation (2) follows from Proposition 5 by continuity of  $f$ , if the limit exists. As continuous quasi-periodicity implies almost periodicity, Proposition 4 proves existence. □

With this result and Proposition 3, one can characterize the frequencies that can occur by the imaginary parts of the eigenvalues of  $A$ .

**Corollary 2.** *Let  $s \in S^{n-1}$  and  $\omega \in \mathbb{R}$ . Let  $\text{spec}_s A = \{\lambda_1, \dots, \lambda_m\}$ . Let  $Z := \{\omega, \Im\lambda_1, \dots, \Im\lambda_m\} \setminus \{0\}$ . If the numbers  $1/z$ ,  $z \in Z$ , are rationally independent, then  $f_\omega^*(s) = 0$  holds for all continuous  $f$ .*

*Proof.* This follows from Proposition 3 and Proposition 6. Compare [8, Corollary 2.5.17]. □

If  $\Phi_t s$  approaches a periodic orbit, one gets the following stronger result.

**Theorem 4.2.** *Let  $s \in S^{n-1}$  and  $\omega \in \mathbb{R}$ . For a period  $\tau > 0$ , assume that  $Q_t s$  is  $\tau$ -periodic in  $t$ , i. e., that  $\tau/2\pi \cdot \Im\lambda \in \mathbb{Z}$  for all  $\lambda \in \text{spec}_s A$ . Further assume that  $Q_t s$  is not constant in  $t$ , i. e., that there is at least one  $\lambda \in \text{spec}_s A$  with  $\Im\lambda \neq 0$ . If  $\omega\tau/2\pi \notin \mathbb{Z}$ , then  $f_\omega^*(s) = 0$  for all continuous  $f : S^{n-1} \rightarrow \mathbb{C}$ . On the other hand, for every  $k \in \mathbb{Z}$ , there is a continuous function  $f : S^{n-1} \rightarrow \mathbb{C}$  such that  $f_{k\omega}^*(s) \neq 0$ .*

*Proof.* This follows from Proposition 2, Theorem 3.2 and Proposition 6. Compare [8, Theorem 2.5.18]. □

**4.2. Periodic equations.** Consider the linear differential equation

$$\dot{x} = A(t)x \tag{3}$$

in  $\mathbb{C}^n$  for a  $\tau$ -periodic continuous map  $A : \mathbb{R} \rightarrow \mathbb{C}^{n \times n}$ ,  $\tau > 0$ . By Floquet theory, a fundamental solution of this system can be given by  $P(t)e^{Rt}$ , where  $P : \mathbb{R} \rightarrow \mathbb{C}^{n \times n}$  is nonsingular, differentiable and  $\tau$ -periodic, and  $R \in \mathbb{C}^{n \times n}$ . Note that  $P$  and  $R$  are not uniquely determined. Eigenvalues of  $R$  are called Floquet exponents.

We project the solution of (3) onto the unit sphere  $S^{n-1} \subset \mathbb{C}^n$ . This yields the map  $\Phi : \mathbb{R} \times S^{n-1} \rightarrow S^{n-1}$ ,  $(t, s) \mapsto \Phi_t s$  with

$$\Phi_t s := \frac{P(t)e^{Rt}s}{\|P(t)e^{Rt}s\|},$$

where  $\|\cdot\|$  denotes the Euclidean norm. This map is no semi-flow, so recall Remark 1 and Remark 2.

For every starting point  $s \in S^{n-1}$ , the map  $\Phi_t s$  asymptotically approaches a quasi-periodic orbit  $Q_t s$  with periods given by  $2\pi/\tau$  and  $2\pi/\Im\lambda$ ,  $\lambda \in \text{spec}_s R$ , where  $\text{spec}_s R$  is a subset of  $\text{spec} R$  determined by the point  $s$ . Note that  $Q_t s$  is continuous in  $t$ . For more details see [8, Section 2.5]. Note that this  $Q_t$  does not coincide with that from the previous section 4.1.

**Proposition 7.** *Let  $A : \mathbb{R} \rightarrow \mathbb{C}^{n \times n}$  be continuous and  $\tau$ -periodic,  $\tau > 0$ , and consider  $\Phi$ . Let  $s \in S^{n-1}$  and  $\omega \in \mathbb{R}$ . Then  $f_\omega^*(s)$  exists for every continuous  $f : S^{n-1} \rightarrow \mathbb{C}$ , and for any Floquet representation by  $P$  and  $R$ , it holds that*

$$f_\omega^*(s) = \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T e^{i\omega t} f \left( \frac{P(t)q(R, s, t)}{\|P(t)q(R, s, t)\|} \right) dt. \tag{4}$$

*Proof.* By [8, Lemma 2.5.28], it holds that  $\Phi_t s - Q_t s \rightarrow 0$  for  $t \rightarrow \infty$ . So Equation (4) follows from Proposition 5 by continuity of  $f$ , if the limit exists. As continuous quasi-periodicity implies almost periodicity, Proposition 4 proves existence. □

With this result and Proposition 3, one can characterize the frequencies that can occur by  $\tau$  and the imaginary parts of the Floquet exponents.

**Corollary 3.** *Let a Floquet representation by  $P$  and  $R$  be given. Let  $s \in S^{n-1}$  and  $\omega \in \mathbb{R}$ . Let  $\text{spec}_s R = \{\lambda_1, \dots, \lambda_m\}$ . Let  $Z := \{\omega, 2\pi/\tau, \Im\lambda_1, \dots, \Im\lambda_m\} \setminus \{0\}$ . If the numbers  $1/z$ ,  $z \in Z$ , are rationally independent, then  $f_\omega^*(s) = 0$  holds for all continuous  $f$ .*

*Proof.* This follows from Proposition 3 and Proposition 7. Compare [8, Corollary 2.5.32].  $\square$

If  $\Phi_t s$  approaches a periodic orbit, one gets the following stronger result.

**Theorem 4.3.** *Let  $A : \mathbb{R} \rightarrow \mathbb{C}^{n \times n}$  be continuous and  $\tau$ -periodic,  $\tau > 0$ , and consider the flow  $\Phi$ . Let a fundamental solution be given by  $P(t)e^{Rt}$ , where  $P : \mathbb{R} \rightarrow \mathbb{C}^{n \times n}$  is  $\tau$ -periodic and  $R \in \mathbb{C}^{n \times n}$ . Let  $s \in S^{n-1}$ , and assume that, for every  $\lambda \in \text{spec}_{J_{\max}(s)} R \setminus \mathbb{R}$ , there are coprime numbers  $a_\lambda, b_\lambda \in \mathbb{N}$  such that*

$$a_\lambda \tau = b_\lambda \frac{2\pi}{\Im \lambda}$$

*Let  $a := \text{lcm}\{a_\lambda \mid \lambda \in \text{spec}_{J_{\max}(s)} R \setminus \mathbb{R}\}$ , and  $\sigma := a\tau$ . Then for every  $k \in \mathbb{Z}$ , there is  $f : S^{n-1} \rightarrow \mathbb{R}$  such that  $f_{2k\pi/\sigma}^*(s) \neq 0$ .*

*Proof.* See [8, Theorem 2.5.35]  $\square$

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