DISCRETE AND CONTINUOUS
DYNAMICAL SYSTEMS
Supplement 2013

Website: www.aimSciences.org

pp. **407–414**

ASYMPTOTIC BEHAVIOR OF SOLUTIONS TO A COUPLED SYSTEM OF MAXWELL'S EQUATIONS AND A CONTROLLED DIFFERENTIAL INCLUSION

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Abstract. The present article consists of two parts. In the first part we consider evolutionary variational inequalities with a nonlinearity which is described by a differential inclusion. Using the frequency-domain method we prove, under certain assumptions, the dissipativity of our variational inequality which is important for the asymptotic behavior of the system. In the second part a coupled system of Maxwell's equation and the heat equation is considered. For this system we introduce the notion of stability on a finite-time interval and present a theorem on this type of stability.

1. **Evolutionary variational inequalities.** Suppose that Y_0 is a real Hilbert space with $(\cdot,\cdot)_0$ and $\|\cdot\|_0$ as scalar product resp. norm. Suppose also that $A : \mathcal{D}(A) \subset Y_0 \to Y_0$ is an unbounded densely defined linear operator. The Hilbert space Y_1 is defined as $\mathcal{D}(A)$ equipped with the scalar product

$$(y,\eta)_1 := \left((\beta I - A)y,\, (\beta I - A)\eta\right)_0\,, \quad y,\eta \in \mathcal{D}(A)\,, \tag{1}$$

where $\beta \in \rho(A)\cap\mathbb{R}$ ($\rho(A)$ the resolvent set of A) is an arbitrary but fixed number the existence of which we assume. The Hilbert space Y_{-1} is by definition the completion of Y_0 with respect to the norm $\|y\|_{-1} := \|(\beta I - A)^{-1}y\|_0$. Thus we have the dense and continuous imbedding

$$Y_1 \subset Y_0 \subset Y_{-1} \tag{2}$$

which is called *Hilbert space rigging structure*. In this triple, Y_0 is the *pivot space*, Y_1 is the *interpolation space* and Y_{-1} is the *extrapolation space* ([8]). The *duality pairing* $(\cdot,\cdot)_{-1,1}$ on $Y_{-1}\times Y_1$ is the unique extension by continuity of the functionals $(\cdot,y)_0$ with $y \in Y_1$ onto Y_{-1}. If $T > 0$ is an arbitrary number we define the norm for Bochner measurable functions in $L^2(0,T;Y_j),\ j = 1,0,-1$ through

$$\|y(\cdot)\|_{2,j} := \left(\int_0^T \|y(t)\|_j^2\,dt\right)^{1/2}. \tag{3}$$

2010 *Mathematics Subject Classification.* Primary: 35B35, 35B40; Secondary: 35K15, 35L20, 80A20.

Key words and phrases. Asymptotic behavior, evolutionary variational inequality, stability, dissipativity, frequency-domain condition, Maxwell's equation.

The authors are supported by DAAD and the German-Russian Interdisciplinary Science Center (G-RISC).

Let $\mathcal{W}_T$ be the space of functions $y(\cdot) \in L^2(0,T;Y_1)$ for which $\dot{y}(\cdot) \in L^2(0,T;Y_{-1})$, equipped with the norm

$$\|y(\cdot)\|_{\mathcal{W}_T} := \left(\|y(\cdot)\|_{2,1}^2 + \|\dot{y}(\cdot)\|_{2,-1}^2\right)^{1/2}. \tag{4}$$

Suppose $Y_1 \subset Y_0 \subset Y_{-1}$ is a real Hilbert space rigging structure with $A \in \mathcal{L}(Y_1, Y_{-1})$. Assume that U and W are two real Hilbert spaces with scalar products $(\cdot,\cdot)_U$, $(\cdot,\cdot)_W$ and norms $\|\cdot\|_U, \|\cdot\|_W$, respectively. Introduce the linear continuous operators

$$B : U \to Y_{-1}\ ,\ C : Y_1 \to U\ , \tag{5}$$

and define the set-valued map

$$\varphi : W \to 2^U \tag{6}$$

and the map

$$\psi : Y_1 \to \mathbb{R}_+ \cup \{+\infty\}\,. \tag{7}$$

Note that in applications φ is a *material law nonlinearity*, ψ is a *contact-type* or *friction functional* and $w(t) = Cy(t)$ is the *input* of the nonlinearity. Consider the *evolutionary variational inequality* with set-valued nonlinearity ([1])

$$(\dot{y} - Ay - Bu, \eta - y)_{-1,1} + \psi(\eta) - \psi(y) \geq 0\,, \quad \forall\, \eta \in Y_1\,, \tag{8}$$

$$w(t) = Cy(t)\ ,\ u(t) \in \varphi(w(t))\ ,\ y(0) = y_0 \in Y_0\,. \tag{9}$$

Remark 1. In the contact free case when $\psi \equiv 0$ the evolutionary variational inequality (8), (9) is equivalent to an *evolution equation* with a set-valued nonlinearity φ given by

$$\dot{y} = Ay + Bu \qquad \text{in } \ Y_{-1}\,, \tag{10}$$

$$w(t) = Cy(t)\ ,\ u(t) \in \varphi(w(t))\ ,\ y(0) = y_0 \in Y_0\,. \tag{11}$$

Definition 1.1. A function $y(\cdot) \in \mathcal{W}_T \cap C(0,T;Y_0)$ is said to be a solution of (8), (9) on $(0,T)$ if there exists a function $u(\cdot) \in L^2(0,T;U)$ such that for a.a. $t \in (0,T)$ the inequality (8), (9) is satisfied and $\int_0^T \psi(y(t))dt < +\infty$. The pair $\{y(\cdot), u(\cdot)\}$ is called a **response** of (8), (9); $u(\cdot)$ is an **associated selection**.

Suppose that F, G and H are quadratic forms on $Y_1 \times U$. The *class* $\mathcal{N}(F,G)$ ($\mathcal{N}(F,G,H)$) *of nonlinearities* for (8), (9) consists of all maps (6) such that the following condition **(a)** is satisfied (conditions **(a)** and **(b)** are satisfied):

(a) For any $T > 0$ and any two functions $y(\cdot) \in L^2(0,T;Y_1)$ and $u(\cdot) \in L^2(0,T;U)$ with

$$u(t) \in \varphi(Cy(t)) \quad \text{for a.a.} \quad t \in [0,T]\ , \tag{12}$$

it follows that

$$F(y(t), u(t)) \geq 0 \quad \text{for a.a.} \quad t \in [0,T]\ , \tag{13}$$

and there exists a continuous function $\Phi : Y_1 \to \mathbb{R}_+$ (*generalized potential*) and numbers $\lambda > 0$ and $\gamma > 0$ such that

$$\int_s^t G(y(\tau), u(\tau))d\tau \geq \frac{1}{2}(\Phi(y(t)) - \Phi(y(s))) + \lambda \int_s^t \Phi(y(\tau))d\tau \tag{14}$$

for all $0 \leq s < t \leq T$ and

$$\Phi(y) \geq \gamma \|y\|_0^2, \quad \forall\, y \in Y_0\ . \tag{15}$$

.

(b) For any $T > 0$ and any two pairs of functions $y_1(\cdot)$, $y_2(\cdot) \in L^2(0,T;Y_1)$ and $u_1(\cdot)$, $u_2(\cdot) \in L^2(0,T;U)$ with

$$u_i(t) \in \varphi(Cy_i(t)), \quad i = 1,2 \quad \text{for a.a.} \quad t \in [0,T] \,, \tag{16}$$

it follows that

$$H(y_1(t) - y_2(t), u_1(t) - u_2(t)) \geq 0 \quad \text{for a.a.} \quad t \in [0,T] \,. \tag{17}$$

(A1) For fixed linear operators A, B, C, fixed function (7), arbitrary $y_0 \in Y_0$, $T > 0$ and $\varphi \in \mathcal{N}(F,G,H)$ ($\varphi \in \mathcal{N}(F,G)$) there exists a response $\{y(\cdot), u(\cdot)\}$ of (8), (9).

2. **Observations that are determining for the dissipativity.** Suppose S is a real Hilbert space (*observation space*), $M : Y_1 \to S$ is a given linear bounded operator (*observation operator*), $P \in \mathcal{L}(Y_{-1}, Y_0) \cap \mathcal{L}(Y_0, Y_1)$, $P = P^*$ in Y_0, is also given such that the following conditions are satisfied:

(1) $V_1(y) := \frac{1}{2}(y, Py)_0 \geq 0 \,, \quad \forall\, y \in Y_0$;

(2) $V(y) := V_1(y) + \frac{1}{2}\Phi(y) \geq \text{const} \cdot \|y\|_0^2, \quad \forall\, y \in Y_0$;

(3) There exist numbers $\lambda > 0$ and $\mu > 0$ such that for an arbitrary solution $y(\cdot)$ of (8), (9) the function $m(t) := V(y(t))$ satisfies

$$\dot{m}(t) + 2\lambda m(t) + \psi(y(t)) - \psi(-Py(t) + y(t)) \leq \mu \|My(t)\|_S^2 \quad \text{for a.a.} \quad t \geq 0 \,. \tag{18}$$

Then the observation

$$\sigma(t) := \mu \|My(t)\|_S^2 \tag{19}$$

is *determining for the dissipativity with domain* $\mathcal{D}$ of (8), (9), i.e., the property

$$\int_t^{t+1} \|My(\tau)\|_S^2 d\tau \to 0 \quad \text{for} \quad t \to +\infty$$

implies that $\lim_{t\to+\infty} \sup m(t) \leq C$ and, consequently, (8), (9) is (point) *dissipative* with *domain of dissipativity*

$$\mathcal{D} := \{y \in Y_0 : \|y\|_0^2 \leq \frac{2C}{\gamma}\} \,, \tag{20}$$

where $\gamma > 0$ is from (15).

3. **Frequency-domain conditions for the existence of determining observations.** We consider the existence problem for observations that are determining for dissipativity. Our goal is to find effective conditions for the existence of Lyapunov-type functions V. A general approach consists in using the *Frequency Theorem* which is also called *Kalman-Yakubovich-Popov Lemma* (KYP Lemma ([4]). Let us state the assumptions for this theorem.

(A2) There exists a number $\lambda > 0$ such that for any $T > 0$ and any $f \in L^2(0,T;Y_{-1})$ the problem

$$\dot{y} = (A + \lambda I)y + f(t) \,,\; y(0) = y_0 \tag{21}$$

is *well-posed*, i.e., for arbitrary $y_0 \in Y_0$, $f(\cdot) \in L^2(0,T;Y_{-1})$ there exists an unique solution $y(\cdot) \in \mathcal{W}_T$ satisfying (21) in the sense that

$$(\dot{y}, \eta)_{-1,1} = ((A + \lambda I)y, \eta)_{-1,1} + (f(t), \eta)_{-1,1} \,, \quad \forall\, \eta \in Y_1 \,, \quad \text{for a.a.} \quad t \in [0,T] \,, \tag{22}$$

and depending continuously on the initial data, i.e.,

$$\|y(\cdot)\|_{\mathcal{W}_T}^2 \leq c_1 \|y_0\|_0^2 + c_2 \|f(\cdot)\|_{2,-1}^2 \,, \tag{23}$$

where $c_1 > 0$ and $c_2 > 0$ are some constants. Furthermore, any solution of

$$\dot{y} = (A + \lambda I)y\ ,\ y(0) = y_0 \tag{24}$$

is exponentially decreasing for $t \to +\infty$, i.e., there exist constants $c_3 > 0$ and $\epsilon > 0$ such that

$$\|y(t)\|_0 \le c_3 e^{-\epsilon t}\|y_0\|_0\ , \quad t > 0\,. \tag{25}$$

(A3) There exists a number $\lambda > 0$ such that the operator $A + \lambda I \in \mathcal{L}(Y_1, Y_{-1})$ is *regular*, i.e., for any $T > 0$, $y_0 \in Y_1$, $z_T \in Y_1$ and $f \in L^2(0, T; Y_0)$ the solutions of the *direct problem*

$$\dot{y} = (A + \lambda I)y + f(t)\ ,\ y(0) = y_0\ , \quad \text{for a.a.} \quad t \in [0, T]\ , \tag{26}$$

and of the *dual problem*

$$\dot{z} = -(A + \lambda I)^* z + f(t)\ ,\ z(0) = z_T\ , \quad \text{for a.a.} \quad t \in [0, T]\ , \tag{27}$$

are strongly continuous in t in the norm of Y_1.

In the next assumption which is called *frequency-domain condition* it is necessary to consider the *complexification* of spaces and linear operators under consideration.

The elements of the complexification Y_0^c of the real Hilbert space Y_0 can be written as $x + iy$ with $x,\ y \in Y_0$ and the inner product of Y_0^c will be denoted by $(\cdot,\cdot)_{Y_0^c}$. The complexification of the other spaces is defined in a similar way. For the linear operator $A : Y_1 \to Y_{-1}$ we denote by A^c the linear operator $A^c : Y_1^c \to Y_{-1}^c$ defined by $A^c(x + iy) = Ax + iAy$. Again, the complexification of the other linear operators which will appear below is defined in a similar way.

Consider now the complexification of the quadratic form F (similarly of G). Suppose that

$$F(y, u) = (F_1 y, y)_{-1,1} + 2(F_2 y, u)_U + (F_3 u, u)_U \tag{28}$$

for $(y, u) \in Y_1 \times U$, where $F_1 = F_1^* \in \mathcal{L}(Y_1, Y_{-1})$, $F_2 \in \mathcal{L}(Y_1, U)$ and $F_3 = F_3^* \in \mathcal{L}(U, U)$. The complexification of the quadratic form (28) is the Hermitian form F^c defined on $Y_1^c \times U^c$ by

$$F^c(y, u) = (F_1^c y, y)_{Y_{-1}^c, Y_1^c} + 2\mathrm{Re}(F_2^c y, u)_{U^c} + (F_3^c u, u)_{U^c}\,. \tag{29}$$

(A4) (Frequency-domain condition) There exist numbers $\lambda > 0$ and $\mu > 0$ such that the following two properties are hold.

a) $F^c(y, u) + G^c(y, u) - \mu\|M^c y\|^2_{S^c} \le 0$ (30)
$\forall\ (y, u) \in Y_1^c \times U^c : \exists \omega \in \mathbb{R} \quad \text{with} \quad i\omega y = (A^c + \lambda I^c)y + B^c u\,.$

b) The functional

$$J(y(\cdot), u(\cdot)) := \int_0^\infty [F^c(y(\tau), u(\tau)) + G^c(y(\tau), u(\tau)) - \mu\|M^c y(\tau)\|^2_{S^c}]d\tau \tag{31}$$

is bounded from above on the set $\mathcal{M}_{y_0} :=$

$$\{y(\cdot), u(\cdot) : \dot{y} = (A^c + \lambda I^c)y + B^c u\,,\ y(0) = y_0\,,\ y(\cdot) \in \mathcal{W}^c_\infty\,,\ u(\cdot) \in L^2(0, \infty; U^c)\}$$

for any $y_0 \in Y_0^c$, i.e., for any such y_0 there exists a $\gamma(y_0) \in \mathbb{R}$ such that $J(y(\cdot), u(\cdot)) \le \gamma(y_0)$.

Theorem 3.1. *Suppose that there exist numbers $\lambda > 0$ and $\delta > 0$ such that the assumptions* **(A1) - (A4)** *are satisfied for (6) - (9) with $\varphi \in \mathcal{N}(F, G)$ and an observation given by (19). Then the observation (19) is determining for the dissipativity of (8), (9) with domain $\mathcal{D}$ given by (20).*

Idea of the proof: We try to find an operator $P = P^* \in \mathcal{L}(Y_{-1}, Y_0) \cap \mathcal{L}(Y_0, Y_1)$ with $(y, Py)_0 \geq 0$, $\forall\, y \in Y_0$ and numbers $\lambda > 0$, $\mu > 0$ such that for any solution $y(\cdot)$ of (8), (9) and their associated generalized potential Φ from condition (14) the integrated inequality (18) is true on any time interval $0 < s < t$, i.e.,

$$m(t) - m(s) + 2\lambda \int_s^t m(\tau) d\tau + \int_s^t p(\tau) d\tau \leq \int_s^t g(\tau) d\tau \,. \tag{32}$$

In (32) we have introduced the functions

$$m(t) := \frac{1}{2}(y(t), Py(t))_0 + \frac{1}{2}\Phi(y(t)) \,, \tag{33}$$

$$p(t) := \psi(y(t)) - \psi(y(t) - Py(t)) \,, \tag{34}$$

$$g(t) := -\mu \|My(t)\|_S^2 \,. \tag{35}$$

In order to guarantee the inequality (32) we choose an operator $P = P^* \in \mathcal{L}(Y_{-1}, Y_0) \cap \mathcal{L}(Y_0, Y_1)$ and numbers $\lambda > 0$, $\mu > 0$ such that

$$(-(A+\lambda I)v - B\zeta, Pv)_{-1,1} \geq F(v,\zeta) + G(v,\zeta) - \mu\|Mv\|_S^2 \,,\ \forall\, v \in Y_1 \,,\ \forall\, \zeta \in U \,. \tag{36}$$

The existence of such P with $(y, Py)_0 \geq 0$, $\forall\, y \in Y_0$ follows due to the assumptions **(A1) - (A4)** from the infinite-dimensional version of the Kalman-Yakubovich-Popov Lemma. From (8), (9) it follows with $v := y(t)$ and $\zeta := u(t)$ that

$$\begin{aligned}&(\dot{y}(t), Py(t))_{-1,1} + \lambda(y(t), Py(t))_0 - ((A + \lambda I)y(t) + \\ &Bu(t), Py(t))_{-1,1} + p(t) \leq 0 \,, \text{a.a. } t > 0 \,.\end{aligned} \tag{37}$$

Using the estimate (36) we derive from (37) the inequality

$$\begin{aligned}(\dot{y}(t), Py(t))_{-1,1} + \lambda(y(t), Py(t))_0 + F(y(t), u(t)) + G\,(y(t), u(t)) \\ - \mu\|My(t)\|_S^2 + p(t) \leq 0 \,, \quad \text{a.a. } t > 0 \,.\end{aligned} \tag{38}$$

Integration of (38) on the time interval $0 < s < t$ gives

$$\begin{aligned}\frac{1}{2}(y(t), Py(t))_0 - \frac{1}{2}(y(s), Py(s))_0 + \lambda \int_s^t (y(\tau), Py(\tau))_0 d\tau + \int_s^t F(y(\tau), u(\tau)) d\tau \\ + \int_s^t G(y(\tau), u(\tau)) d\tau + \int_s^t p(\tau) d\tau \leq \mu \int_s^t \|My(t)\|_S^2 d\tau \,.\end{aligned} \tag{39}$$

From the inequalities (13) and (14) it follows that

$$\int_s^t F(y(\tau), u(\tau)) d\tau \geq 0 \tag{40}$$

and

$$\int_s^t G(y(\tau), u(\tau)) d\tau \geq \frac{1}{2}[\Phi(y(t)) - \Phi(y(s))] + \lambda \int_s^t \Phi(y(\tau)) d\tau \,, \quad 0 < s < t \,. \tag{41}$$

Taking into account now (39) - (41) we obtain that

$$\begin{aligned}&\frac{1}{2}(y(t), Py(t))_0 + \frac{1}{2}\Phi(y(t)) - \frac{1}{2}(y(s), Py(s))_0 - \frac{1}{2}\Phi(y(s)) \\ &+2\lambda \int_s^t [\frac{1}{2}(y(\tau), Py(\tau))_0 - \frac{1}{2}\Phi(y(\tau))] d\tau + \int_s^t p(\tau) d\tau \leq \mu \int_s^t \|My(\tau)\|_S^2\, d\tau \,.\end{aligned} \tag{42}$$

Now we conclude that (42) implies the inequality (32) with the functions $m(\cdot)$, $p(\cdot)$ and $g(\cdot)$ defined by (33) - (35).

Remark 2. The frequency-domain condition **(A4)** depends on imbedding properties of the Sobolev spaces under consideration. Assume, for example, that $G \equiv 0$ and

$$F(y,u) = \beta_0 \|y\|_0^2 - \beta_1 \|y\|_1^2 \,, \quad (y,u) \in Y_0 \times U \,, \tag{43}$$

where β_0 and β_1 are certain real constants.

In order to verify (30) we introduce the frequency-domain characteristic

$$\chi(i\omega) := (i\omega I^c - A^c_\lambda)^{-1} B^c \tag{44}$$

for $\omega \in \mathbb{R}$ s. t. $i\omega \in \rho(A^c_\lambda)$, where $A^c_\lambda := A^c + \lambda I^c$.

It follows that the frequency-domain condition (30) is satisfied if

$$\beta_0 \|\chi(i\omega)u\|^2_{Y_0^c} - \beta_1 \|\chi(i\omega)u\|^2_{Y_1^c} - \delta \|M^c \chi(i\omega)u\|^2_{S^c} \le 0 \,, \qquad \forall\, u \in U^c \,, \quad \forall\, \omega \in \mathbb{R} : \quad i\omega \in \rho(A^c_\lambda) \,. \tag{45}$$

Suppose now that from the imbedding $Y_1^c \subset Y_0^c \subset Y_{-1}^c$ and the properties of the observation operator M we have the a priori estimate

$$\|v\|^2_{Y_0^c} \le c_1 \|v\|^2_{Y_1^c} + c_2 \varepsilon_{M^c} \|M^c v\|^2_{S^c} \,, \quad \forall\, v \in Y_1^c \,, \tag{46}$$

where $c_1 > 0$ and $c_2 > 0$ are certain constants and

$$\varepsilon_{M^c} = \varepsilon_{M^c}(Y_1^c, Y_0^c) := \sup \{ \|w\|_{Y_0^c} : w \in Y_1^c \,,\, M^c w = 0_{S^c} \,, \|w\|_{Y_1^c} \le 1 \} \tag{47}$$

is the *completeness defect* of the observation operator M^c with respect to the imbedding $Y_1^c \subset Y_0^c$.

It follows from (46) that the frequency-domain condition (30) is satisfied if

$$\beta_0\, c_1 \|\chi(i\omega)u\|^2_{Y_1^c} - \beta_1 \|\chi(i\omega)u\|^2_{Y_1^c} + \beta_0 c_2 \varepsilon_{M^c} \|M^c \chi(i\omega)u\|^2_{S^c} - \mu \|M^c \chi(i\omega)u\|^2_{S^c} \le 0 \qquad \forall\, u \in U^c \,, \quad \forall\, \omega \in \mathbb{R} : \quad i\omega \in \rho(A^c_\lambda) \,. \tag{48}$$

For (48) it is sufficient that

$$\beta_0\, c_1 - \beta_1 \le 0 \quad \text{and} \quad \beta_0\, c_2\, \varepsilon_{M^c} - \delta \le 0 \,. \tag{49}$$

We see that if $\beta_0 c_1 - \beta_1 \le 0$ the second condition of (49) is always satisfied if the completeness defect of the observation operator is small. In this case, assuming that the other assumptions for the Theorem 3.1 are also satisfied, it follows that the observation $\sigma(t) = My(t)$ is determining for the dissipativity .

Suppose that $M_k y := (l_1(y), \ldots, l_k(y))$, where $l_i : Y_1 \to \mathbb{R},\ i = 1, \ldots, k$, are continuous linear functionals and $Y_1 = W^{s,2}(\Omega), Y_0 = W^{\sigma,2}(\Omega)$ with $s > \sigma$. Then $\varepsilon_{M^c} \approx c_1 (\frac{c_2}{k})^{s-\sigma}$, i.e., the completeness defect of the observation operator M_k depends on the smoothness properties of the imbedding $Y_1^c \subset Y_0^c$ ([8]).

4. **Maxwell's equations in one-space dimension with thermal effect.** Let us consider the initial-boundary problem

$$\begin{array}{ll} w_{tt} - w_{xx} + \sigma(\theta) w_t = 0, & x \in (0,1), t \in (0,T), \\ A(\theta)_t - \theta_{xx} \ni \sigma(\theta) {w_t}^2, & x \in (0,1), t \in (0,T), \\ w(0,t) = u_1(t), w(1,t) = u_2(t), & t \in (0,T), \\ \theta(0,t) = \theta(1,t) = 0, & t \in (0,T), \\ \theta(x,0) = \theta_0(x), & x \in (0,1), \\ w(x,0) = w_0(x),\ w_t(x,0) = w_1(x), & x \in (0,1), \end{array} \tag{50}$$

where $\theta(x,t)$ is the temperature, $w(x,t)$ is the composite variable, describing the effect of electric and magnetic fields, $\sigma = \sigma(\theta)$ is the dielectric conductivity, which depends on the temperature. The heat equation is understood as inclusion, where $A(\theta)$ is the enthalpy operator, which will be considered as follows

$$A(\theta) = \begin{cases} \theta - 1, & \text{if} \quad \theta < m, \\ [m-1, m], & \text{if} \quad \theta = m, \\ \theta, & \text{if} \quad \theta > m, \end{cases} \tag{51}$$

where m is the melting temperature. Denote $Q_T = (0,1) \times (0,T)$. We see that if $\{(x,t) \in Q_T : \theta(x,t) = m\}$ is a set of measure zero in Q_T, then the differential inclusion in (50) can be considered as differential equation

$$A(\theta)_t - \theta_{xx} = \sigma(\theta) {w_t}^2, \quad x \in (0,1), t \in (0,T) \tag{52}$$

a.e. on Q_T.

System (50) - (51), including the operator $A(\theta)$, was considered in [5].

Suppose that the following conditions hold:

(A1.1) There exists a constant $C_\sigma > 0$ such that $0 \leq \sigma(\theta) \leq C_\sigma(1+\theta) \quad \forall\, \theta \geq 0\,;$ σ satisfies a local Lipschitz condition on $(0, +\infty)$.

(A1.2) There exists a constant $a_0 > 0$ such that $A'(\theta) \geq a_0 \quad$ a.a. $\theta \in \mathbb{R}$.

(A1.3) $u_1, u_2 \in H^1(0,T), \quad u_1(0) = 0, u_2(0) = 0, \quad w_1(x), \theta_0(x) \in L^2(0,1)$.

In [5] there was shown that under the above assumptions there exists a weak solution $(w(x,t), \theta(x,t))$ of problem (50) for any $T < +\infty$; moreover, $w \in C([0,T];$ $L^2(0,1))$; $\theta \in L^2(0,T; H^1(0,1)) \cap C([0,T]; L^2(0,1))$.

In the sequel we assume that $A(\theta)$ is the identity operator. Let us reduce the problem (50) to a problem with homogeneous boundary conditions. Set $u(x,t) = u_1(t)(1-x) + u_2(t)x$ and make the change of variables $W(x,t) = w(x,t) - u(x,t)$, $V(x,t) = W_t(x,t) + u_t(x,t)$. We obtain the system

$$\begin{array}{ll} V - W_t = u_t, & x \in (0,1), t \in (0,T) \\ V_t - W_{xx} + \sigma(\theta) W_t = -u_{tt} - \sigma(\theta) u_t, & x \in (0,1), t \in (0,T), \\ \theta_t - \theta_{xx} = \sigma(\theta) V^2, & x \in (0,1), t \in (0,T), \end{array} \tag{53}$$

with initial and boundary conditions

$$W(0,t) = W(1,t) = 0, \quad \theta\,(0,t) = \theta\,(1,t) = 0, \quad t \in (0,T), \tag{54}$$

$$\begin{array}{ll} W(x,0) = W_0(x) = w_0(x) - u(x,0), & x \in (0,1), \\ W_t(x,0) = W_1(x) = w_1(x) - u_t(x,0), & x \in (0,1), \\ \theta\,(x,0) = \theta_0\,(x)\,, & x \in (0,1). \end{array} \tag{55}$$

Define the space $X = H_1^0(0,1) \times L^2(0,1) \times L^2(0,1)$ with the norm

$$\|(w,v,\theta)\|_X^2 = \|w_x\|_{L^2(0,1)}^2 + \|v\|_{L^2(0,1)}^2 + \|\theta\|_{L^1(0,1)}^2.$$

Let us define the function $y(t) = (W(\cdot,t), V(\cdot,t), \theta(\cdot,t))$ as solution of the problem (53) – (55) with norm

$$\|y(t)\| = \|W_x\|_{L^2(0,1)}^2 + \|V\|_{L^2(0,1)}^2 + \|\theta\|_{L^1(0,1)}^2, \tag{56}$$

where $W(\cdot,t), V(\cdot,t), \theta(\cdot,t)$ satisfy the system (53) – (55). In the previous paper [2] the notion of stablity on a finite time interval for differential inequalities was introduced. Here we give the definition of finite time stability for differential equations based on the definition from [9].

Definition 4.1. The system (53) - (55) is stable with respect to $(\alpha, \beta, t_0, T', \|\cdot\|)$, $0 < \alpha \leq \beta$, $t_0 \geq 0$ and $T' \geq 0$ are arbitrary, if for every solution $y(t)$, the inequality $\|y(t_0)\| < \alpha$ implies that $\|y(t)\| < \beta$ for all $t \in [t_0, t_0 + T)$.

Theorem 4.2. *Consider the problem (53) - (55). Let the following conditions are satisfied:*

$$(i) \qquad 0 < \sigma_0 \le \sigma(\theta), \quad \forall \theta \ge 0 \,, \tag{57}$$

where σ_0 is some constant,

$$(ii) \qquad |M(t)| \le C_M \; for\; any\; t \in [0, T')\,, \tag{58}$$

where

$$M(t) := \int_0^t \sum_{i=1}^{2} [u_{it} + |u_{itt}|] d\tau. \tag{59}$$

Then the problem (53) - (55) is $(\alpha, \beta, t_0, T', \|\cdot\|)$-stable *with $t_0 = 0$ and for given $\beta > 0$ the number α can be calculated from the following equation*

$$\alpha = \frac{\frac{\beta}{2} - 4\max[C_\sigma, \frac{1}{\sigma_0}]C_M C_{Gr} - 2(C(\beta, T') + 1)C_\sigma C_{Gr} C_M C(f, T')}{C_{Gr}(1 + 2C_{Gr}C_\sigma(C(\beta, T') + 1))}, \tag{60}$$

where $C_\sigma > 0$ is from (A1.1), $f(t) := \sum_{i=1}^{2} |u_{it}|$,

$$C(\beta, T') = \frac{\beta}{2\max[1, \frac{1}{\sigma_0}](C_M + C(f, T'))}, \tag{61}$$

$$C(f, T') = e^{\int_0^{T'} f(\tau) d\tau} \int_0^{T'} f(\tau) e^{-\int_0^{\tau} f(\xi) d\xi} d\tau, \tag{62}$$

C_{Gr} is an upper bound of the solution of the heat equation with homogeneous Dirichlet boundary conditions for any $t \in (0, T')$.

In the proof we use general results from [9] and properties of the one-dimensional model, derived in [3].

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Received for publication September 2012.

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