DISCRETE AND CONTINUOUS
DYNAMICAL SYSTEMS
Supplement 2013

Website: www.aimSciences.org

pp. **653–662**

DYNAMICALLY CONSISTENT DISCRETE-TIME SI AND SIS EPIDEMIC MODELS

Lih-Ing W. Roeger

Department of Mathematics and Statistics
Texas Tech University
Lubbock, TX 79409-1042, USA

Abstract. Discrete-time SI and SIS epidemic models are constructed by applying the nonstandard finite difference (NSFD) schemes to the differential equation models. The difference equation systems are dynamically consistent with their analog continuous-time models. The basic standard incidence SI and SIS models without births and deaths, with births and deaths, and with immigrations, are considered. The continuous models possess either the conservation law that the total population is a constant or the total population N satisfies $N'(t) = \lambda - \mu N$ and so that N approaches a constant λ/μ as t approaches infinity. The difference equation systems via NSFD schemes preserve all properties including the positivity of solutions, the conservation law, and the local and some of the global stability of the equilibria. They are said to be dynamically consistent with the continuous models with respect to these properties. We show that a simple criterion for choosing a certain NSFD scheme such that the positivity solutions are preserved is usually an indication of an appropriate NSFD scheme.

1. **Introduction.**

In many differential equation systems, numerical solutions are necessary for solving and understanding the systems. Numerical solutions, especially finite difference schemes, produce difference equations. If the difference equations preserve certain property P of the differential equations, then it is said that the difference equations are dynamically consistent with the differential equations [9, 11] with respect to property P. The author [16, 17] had shown that for an autonomous differential equation with the right-hand side function as a polynomial, a difference equation produced by some NSFD scheme not only preserves the positivity of solutions, it preserves the positively invariant intervals, the monotonicity of the solutions within each intervals, and the local and global stability of all of the equilibria. For the 2-dimensional case, the author [18] had shown that the NSFD finite difference equation system of a Lotka-Volterra competition system preserves all of the properties of the continuous-time system.

Differential equations are used to model many population, ecology, and epidemic phenomena. Non-negativity of variables is always an essential property of these differential equations. SIS and SIR models are the basic epidemic models studied by many researchers [7]. Mickens and others [4, 12, 13, 14] provided some insights into the NSFD schemes that preserve the conservation law and the positivity of solutions for some epidemic models.

2010 *Mathematics Subject Classification.* 34-04, 39-04, 92-04.

Key words and phrases. Discrete-time, difference equations, SIS, epidemic models, NSFD schemes.

For a first-order system of differential equations

$$\frac{dy}{dt} = f(y); \; y(t_0) = y_0, \tag{1}$$

where $y = y(t)$ is a vector function, we assume a numerical solution is defined as a system of finite difference equations:

$$D_h y_k = F_h(f; y_k), \tag{2}$$

where $D_h y_k$ approximates the derivative dy/dt of the solution $y(t)$, $F_h(f; y_k)$ approximates $f(y(t))$, $h = \Delta t$, $t_k = hk$, and $y_k = y(t_k)$. The finite difference scheme (2) is an NSFD scheme [3, 10] if (i) in the discrete derivative $D_h y_k$, the traditional denominator h is replaced by a nonnegative function $\phi(h)$ such that $\phi(h) = h + O(h^2)$ as $h \to 0$; or (ii) the terms in $f(y)$ are approximated in a nonlocal way. For example, $y^2(t) \approx y_{k+1} y_k$ and $y^3(t) \approx 2y_{k+1}^2 y_k^2/(y_{k+1} + y_k)$.

In this manuscript, we study six basic discrete SI and SIS epidemic models that are constructed by using NSFD or exact finite difference schemes. The SI and SIS models have one common feature: the disease does not contribute to the additional death. The goals are (a) to find exact finite difference schemes that have the forms corresponding to the differential equations if the exact schemes exist and (b) to form NSFD schemes such that the derived difference equations are dynamically consistent with the differential equations. The exact finite difference schemes can be found for the SI and SIS models without immigration because the explicit analytic solutions of these models exist. The exact schemes provide the insight into constructing effective NSFD schemes. If the exact finite difference schemes do not exist, we construct appropriate NSFD schemes. We show that NSFD schemes not only preserve the positivity of solutions, the conservation laws, but also the stability conditions of all of the equilibria solutions.

SI epidemic models are used to model diseases such as influenza or the common cold, where generally no one is immune and over the course of the epidemic almost everyone eventually becomes infected. SIS epidemic model has been applied to some sexually transmitted diseases, where there is recovery but no immunity; individuals can become infected immediately following recovery [2].

Let $S(t)$ and $I(t)$ represent the susceptible and the infectious (or infected) populations at time t. The total population is $N(t) = S(t) + I(t)$. The models to be discussed are as follows: SI model without births and deaths:

$$\begin{aligned} \frac{dS}{dt} &= -\frac{\beta}{N} SI, \\ \frac{dI}{dt} &= \frac{\beta}{N} SI; \end{aligned} \tag{3}$$

SI model with births and deaths:

$$\begin{aligned} \frac{dS}{dt} &= \mu N - \frac{\beta}{N} SI - \mu S, \\ \frac{dI}{dt} &= \frac{\beta}{N} SI - \mu I; \end{aligned} \tag{4}$$

SI model with immigration:

$$\begin{aligned} \frac{dS}{dt} &= \lambda - \frac{\beta}{N} SI - \mu S, \\ \frac{dI}{dt} &= \frac{\beta}{N} SI - \mu I; \end{aligned} \tag{5}$$

SIS model without births and deaths:

$$\begin{aligned} \frac{dS}{dt} &= -\frac{\beta}{N}SI + \gamma I, \\ \frac{dI}{dt} &= \frac{\beta}{N}SI - \gamma I; \end{aligned} \tag{6}$$

SIS model with births and deaths:

$$\begin{aligned} \frac{dS}{dt} &= \mu N - \frac{\beta}{N}SI + \gamma I - \mu S, \\ \frac{dI}{dt} &= \frac{\beta}{N}SI - \gamma I - \mu I; \end{aligned} \tag{7}$$

and SIS model with immigration:

$$\begin{aligned} \frac{dS}{dt} &= \lambda - \frac{\beta}{N}SI + \gamma I - \mu S, \\ \frac{dI}{dt} &= \frac{\beta}{N}SI - \gamma I - \mu I. \end{aligned} \tag{8}$$

The initial conditions are $S(0) = S_0 \geq 0$ and $I(0) = I_0 \geq 0$ for all six models.

The manuscript is organized as follows. The SI models are presented in Section 2. The SIS models are presented in Section 3. The results of each continuous models are outlined then the results of the discrete models are shown and compared to the continuous models. The exact finite difference schemes can be found for the SI and SIS models, (3), (4), (6), and (7). For the SI and SIS models with immigration, (5) and (8), the approximating NSFD schemes are constructed. Conclusion and discussion are presented in Section 4.

2. SI models.

2.1. Simple SI model (3).

Consider the simple SI model (3). The total population $S(t) + I(t) = N$ is a constant. The solution paths $(S(t), I(t))$, $t > 0$, are non-negative. On the SI phase plane, the line segment $S(t) + I(t) = N$ in the first quadrant is positively invariant. Since $S'(t) < 0$, $S(t)$ monotonically decreases with time t and eventually $S(t) \to 0$ and $I(t) \to N$. The disease-free equilibrium $(N, 0)$ is unstable and the endemic equilibrium $(0, N)$ is globally stable. Eventually everyone becomes infected.

Discrete-time versions of the SI model (3) can be constructed by finite difference numerical schemes. For example, with forward Euler scheme we obtain the difference equation system:

$$\begin{aligned} \frac{S_{k+1} - S_k}{h} &= -\frac{\beta}{N}S_k I_k, \\ \frac{I_{k+1} - I_k}{h} &= \frac{\beta}{N}S_k I_k, \end{aligned} \tag{9}$$

where h is the time step-size, $S_k = S(kh)$, and $I_k = I(kh)$ for $k = 0, 1, 2, \ldots$. This model was studied by Allen [1] as a discrete-time SI model in comparison to the continuous SI model (3). To ensure that the solutions are always non-negative, restrictions must be put on the parameter h. More specifically, only under the restriction $h \leq 1/\beta$, the discrete-time system (9) preserves the positivity of the solutions, the global stability of the endemic equilibrium $(0, N)$, and the conservative law $S_k + I_k = N$. With NSFD schemes, it is possible to construct discrete-time models with the above properties with less restriction. To construct an NSFD scheme more effectively, we start with the discussion of the exact finite difference scheme.

Consider the continuous SI model (3). Since $I(t)+S(t) = N =$ constant, system (3) can be reduced to an equation for I, $I'(t) = \beta I(1 - I/N)$. The solution to this equation is $I(t) = \frac{NI_0}{I_0+(N-I_0)e^{-\beta t}}$. This equation has a corresponding exact finite difference scheme [10]:

$$\frac{I_{k+1} - I_k}{\phi} = \beta I_k \left(1 - \frac{I_{k+1}}{N}\right), \text{ where } \phi = \frac{e^{h\beta} - 1}{\beta}.$$

Then the continuous SI model (3) can be written as a system of exact finite difference equations as the following:

$$\begin{aligned} \frac{S_{k+1} - S_k}{\phi} &= -\frac{\beta}{N} S_{k+1} I_k, \\ \frac{I_{k+1} - I_k}{\phi} &= \frac{\beta}{N} S_{k+1} I_k. \end{aligned} \tag{10}$$

Consider the discrete system (10) as an NSFD scheme with $\phi > 0$. The sum of the two equations in (10) is zero. Therefore, the conservative law $S_{k+1} + I_{k+1} = S_k + I_k = N$ always holds. S_{k+1} and I_{k+1} can be solved in terms of S_k and I_k:

$$\begin{aligned} S_{k+1} &= S_k \left(1 - \frac{\beta I_k}{(N/\phi) + \beta I_k}\right), \\ I_{k+1} &= I_k \left(1 + \frac{\beta S_k}{(N/\phi) + \beta I_k}\right). \end{aligned}$$

It is easy to see that both $S_k > 0$ and $I_k > 0$ for all $k = 1, 2, \ldots$ with the initial conditions $S_0 > 0$ and $I_0 > 0$ and $S_k \geq 0$ and $I_k \geq 0$ with the initial conditions $S_0 \geq 0$ and $I_0 \geq 0$. Also, $S_{k+1}/S_k < 1$ and $I_{k+1}/I_k > 1$ for all $k > 0$. Therefore, the sequence $\{S_k\}$ decreases monotonically and the sequence $\{I_k\}$ increases monotonically as k increases. The two equilibria of the discrete system (10) are $(N, 0)$ and $(0, N)$. They are the same equilibria of the continuous-time SI system (3). (S_k, I_k) converges to one of the equilibria. Therefore, $S_k \to 0$ as $k \to \infty$. It concludes that $I_k \to N$ as $k \to \infty$. We have proved the following results.

Theorem 2.1. *The discrete system (10) preserves the properties of the continuous system (3) for any $\phi > 0$. Specifically, if $\phi = (e^{h\beta} - 1)/\beta$, then system (10) is an exact finite difference scheme of the continuous system (3).*

We had purposely chosen the exact or NSFD scheme in the form (10) so that the positivity of solutions are preserved when $\phi > 0$. There are other (equivalent) exact finite difference schemes. For example, the following system

$$\begin{aligned} \frac{S_{k+1} - S_k}{\phi} &= -\frac{\beta}{N} S_k I_{k+1}, \\ \frac{I_{k+1} - I_k}{\phi} &= \frac{\beta}{N} S_k I_{k+1}, \end{aligned} \tag{11}$$

where $\phi = (1 - e^{-h\beta})/\beta$, also expresses an exact finite difference scheme for system (3). The system is so obtained because there is an exact scheme for the equation $I'(t) = \beta I(1 - I/N)$ as

$$\frac{I_{k+1} - I_k}{\phi} = \beta I_{k+1} \left(1 - \frac{I_k}{N}\right), \text{ where } \phi = \frac{1 - e^{-h\beta}}{\beta}.$$

Assuming $\phi > 0$ and solving the discrete system (11) for S_{k+1} and I_{k+1}, we obtain

$$\begin{aligned} S_{k+1} &= S_k \left(1 - \frac{\phi\beta I_k}{N - \phi\beta S_k}\right), \\ I_{k+1} &= I_k \left(\frac{N}{N - \phi\beta S_k}\right). \end{aligned} \tag{12}$$

Consider system (11) as an NSFD scheme for (3) and $\phi > 0$. We can show that the NSFD scheme (11) preserves the positivity of solutions and the global stability of the equilibrium $(0, N)$ if $\phi < 1/\beta$. Both discrete SI systems (10) and (11) are constructed with the assistance of the exact finite difference schemes, but extra restriction is needed for system (11) to be dynamically consistent.

2.2. SI **model with births and deaths (4).** Consider the SI model with births and deaths (4). The total population $N = S(t) + I(t)$ is a constant. The basic reproductive number for the system is $\mathcal{R}_0 = \beta/\mu$. The value $\mathcal{R}_0$ is the number of secondary infections caused by one infectious individual during the individual's infectious period [2]. The equilibria are the disease-free equilibrium $(N, 0)$ and the endemic equilibrium $(N/\mathcal{R}_0, N(1 - 1/\mathcal{R}_0))$. The system can be reduced to $I'(t) = \beta I(1 - I/N) - \mu I$ with solution

$$I(t) = \frac{N(1 - 1/\mathcal{R}_0)}{1 + \left[\frac{N}{I_0}(1 - 1/\mathcal{R}_0) - 1\right] e^{-\mu(\mathcal{R}_0 - 1)t}}.$$

If $\mathcal{R}_0 > 1$, then $I(t) \to N(1 - 1/\mathcal{R}_0)$ as $t \to \infty$; the solutions approach the endemic equilibrium $(N/\mathcal{R}_0, N(1 - 1/\mathcal{R}_0))$. If $\mathcal{R}_0 < 1$, then $I(t) \to 0$; the solutions approach the disease-free equilibrium $(N, 0)$.

The equation for $I(t)$ has an exact finite difference equation

$$\frac{I_{k+1} - I_k}{\phi} = \beta I_k \left(1 - \frac{I_{k+1}}{N}\right) - \mu I_{k+1}, \quad \text{where } \phi = \frac{e^{(\beta-\mu)h} - 1}{\beta - \mu e^{(\beta-\mu)h}}.$$

With this, we can construct an NSFD discrete-time SI system for any $\phi > 0$:

$$\begin{aligned} \frac{S_{k+1} - S_k}{\phi} &= \mu N - \frac{\beta}{N} S_{k+1} I_k - \mu S_{k+1}, \\ \frac{I_{k+1} - I_k}{\phi} &= \frac{\beta}{N} S_{k+1} I_k - \mu I_{k+1}. \end{aligned} \tag{13}$$

In system (13), the conservation law is preserved; $S_k + I_k = N$ a constant. S_{k+1} and I_{k+1} can be solved in terms of S_k and I_k:

$$\begin{aligned} S_{k+1} &= \frac{S_k + \phi\mu N}{1 + \phi\mu + \phi\beta I_k/N}, \\ I_{k+1} &= \frac{(1 + \phi\beta) I_k}{1 + \phi\mu + \phi\beta I_k/N}. \end{aligned} \tag{14}$$

It is easy to see that the positivity of solutions is preserved. The I equation in (14) is a Beverton-Holt equation. Beverton-Holt equation is the following difference equation

$$u_{k+1} = \frac{a u_k}{1 + b u_k}, k = 0, 1, 2, \ldots,$$

with the explicit solution [5]

$$u_k = \frac{(a - 1)u_0}{b u_0 + (a - 1 - b u_0) a^{-k}}, k = 1, 2, \ldots.$$

Therefore, we have the following explicit solution for I_k:

$$I_k = \frac{N(1-1/\mathcal{R}_0)}{1+\left[\frac{N}{I_0}(1-1/\mathcal{R}_0)-1\right]\left(\frac{\phi+1/\mu}{\phi\mathcal{R}_0+1/\mu}\right)^k}.$$

For all $\phi > 0$, it is easy to see that $I_k \to N(1-1/\mathcal{R}_0)$ if $\mathcal{R}_0 > 1$ and $I_k \to 0$ if $\mathcal{R}_0 < 1$. These are the same results as the continuous SI model (4). Therefore, we have the following results.

Theorem 2.2. *The discrete system (13) is dynamically consistent with the continuous SI model with births and deaths (4) for all $\phi > 0$. Specifically, if $\beta \neq \mu$ and h sufficiently small so that $\phi = \frac{e^{(\beta-\mu)h}-1}{\beta-\mu e^{(\beta-\mu)h}} > 0$, then the discrete system is an exact finite difference scheme for system (4).*

2.3. SI **model with immigration (5).** Consider the SI model with immigration (5). The system has a different conservation law, the total population N satisfies $N'(t) = \lambda - \mu N(t)$. The solution for $N(t)$ is $N(t) = \frac{\lambda}{\mu} + \left(N_0 - \frac{\lambda}{\mu}\right)e^{-\mu t}$. The solution $N(t)$ approaches $\frac{\lambda}{\mu}$ as $t \to \infty$. The basic reproductive number for the system is also $\mathcal{R}_0 = \beta/\mu$. There are two equilibria: the disease-free equilibrium $\left(\frac{\lambda}{\mu}, 0\right)$ and the endemic equilibrium $\left(\frac{\lambda}{\beta}, \frac{\lambda}{\beta}(\mathcal{R}_0 - 1)\right)$. If $\mathcal{R}_0 < 1$, the disease-free equilibrium is globally asymptotically stable. If $\mathcal{R}_0 > 1$, then as $t \to \infty$, since $N(t) \to \lambda/\mu$, we have $I'(t) \to \beta I\left(1 - \frac{\mu}{\lambda}I\right) - \mu I$.

The equation $N'(t) = \lambda - \mu N(t)$ has an exact finite difference equation $\frac{N_{k+1}-N_k}{\phi} = \lambda - \mu N_{k+1}$, where $\phi = \frac{e^{\mu h}-1}{\mu}$. With this information, we may consider an NSFD scheme as the following:

$$\begin{aligned} \frac{S_{k+1}-S_k}{\phi} &= \lambda - \frac{\beta}{N_k}S_{k+1}I_k - \mu S_{k+1}, \\ \frac{I_{k+1}-I_k}{\phi} &= \frac{\beta}{N_k}S_{k+1}I_k - \mu I_{k+1}. \end{aligned} \tag{15}$$

This discrete system satisfies the conservation law $(N_{k+1} - N_k)/\phi = \lambda - \mu N_{k+1}$. The explicit solution for N_k is

$$N_k = \frac{\lambda}{\mu} + \left(N_0 - \frac{\lambda}{\mu}\right)(1+\phi\mu)^{-k}, k = 1, 2, \ldots .$$

We see that N_k approaches $\frac{\lambda}{\mu}$ as $k \to \infty$. S_{k+1} and I_{k+1} can be solved in terms of S_k, I_k, and N_k:

$$\begin{aligned} S_{k+1} &= \frac{S_k + \phi\lambda}{1+\phi\mu+\phi\beta I_k/N_k}, \\ I_{k+1} &= \frac{(1+\phi\mu+\phi\beta+\phi^2\beta\lambda/N_k)I_k}{(1+\phi\mu)(1+\phi\mu+\phi\beta I_k/N_k)}. \end{aligned} \tag{16}$$

The above NSFD scheme (16) is an approximation to the continuous-time SI model with immigration. We are not able to find an exact finite difference scheme because there is no explicit solution to the original SI model. The I equation of the above discrete system (16) is not a Beverton-Holt equation. But the asymptotic dynamics of I_k when $k \to \infty$, so $N_k \to \frac{\lambda}{\mu}$ is that of the Beverton-Holt and are the same as the I equation in (14). The solution paths of system (15) behave as if they are the solutions of the discrete-time SI model (13). Therefore, the solutions of the discrete model behave the same as the continuous model for all $\phi > 0$ as $t \to \infty$.

3. The SIS models.

3.1. The simple SIS model without births and deaths (6).

Consider the SIS model (6) where the total population $S(t)+I(t)=N$ is a constant. The basic reproduction number is $\mathcal{R}_0=\beta/\gamma$. If $\mathcal{R}_0<1$, the disease-free equilibrium $(N,0)$ is globally asymptotically stable. If $\mathcal{R}_0>1$, the equilibrium $(N,0)$ is unstable and the endemic equilibrium $(N/\mathcal{R}_0, N(1-1/\mathcal{R}_0))$ is globally asymptotically stable.

Reducing system (6) to an equation for $I(t)$ only, we obtain $I'(t)=\beta I\left(1-\frac{I}{N}\right)-\gamma I$ with explicit solution

$$I(t)=\frac{N(1-1/\mathcal{R}_0)}{1+[\frac{N}{I_0}(1-1/\mathcal{R}_0)-1]e^{-\gamma(\mathcal{R}_0-1)t}}.$$

Hence, one of the exact finite difference schemes for $I(t)$ equation can be written as

$$\frac{I_{k+1}-I_k}{\phi}=\beta I_k\left(1-\frac{I_{k+1}}{N}\right)-\gamma I_{k+1},\quad \text{where } \phi=\frac{e^{(\beta-\gamma)h}-1}{\beta-\gamma\, e^{(\beta-\gamma)h}}. \tag{17}$$

With this scheme, we may construct the following NSFD SIS system:

$$\begin{aligned}\frac{S_{k+1}-S_k}{\phi}&=-\frac{\beta}{N}S_{k+1}I_k+\gamma I_{k+1},\\ \frac{I_{k+1}-I_k}{\phi}&=\frac{\beta}{N}S_{k+1}I_k-\gamma I_{k+1},\end{aligned} \tag{18}$$

with $\phi>0$. If $\phi=\frac{e^{(\beta-\gamma)h}-1}{\beta-\gamma\, e^{(\beta-\gamma)h}}$, then (18) becomes an exact finite difference system generated from (17). Similar to the continuous model (6), the total population for the discrete model (18) is a constant: $S_k+I_k=N$; the conservative law $S_k+I_k=N$ holds for all $k>0$ for model (18) for any $\phi>0$. S_{k+1} and I_{k+1} can be written in terms of S_k and I_k:

$$\begin{aligned}S_{k+1}&=\frac{N(\phi\gamma N+S_k)}{N+\phi(\gamma N+\beta I_k)},\\ I_{k+1}&=\frac{N(1+\phi\beta)I_k}{N+\phi(\gamma N+\beta I_k)}.\end{aligned} \tag{19}$$

It is easy to see from the above two equations that if $S_k\geq 0$ and $I_k\geq 0$ than $S_{k+1}\geq 0$ and $I_{k+1}\geq 0$, i.e., the positivity of solutions is independent of the parameters ϕ, β, and γ; the positivity of solutions is preserved.

The I equation in system (19) is a Beverton-Holt equation. Therefore, the explicit solution $I(t)$ for the discrete SIS model can be found as

$$I_k=\frac{N(1-1/\mathcal{R}_0)}{1+\left[\frac{N}{I_0}(1-1/\mathcal{R}_0)-1\right]\left(\frac{\phi+1/\gamma}{\phi\mathcal{R}_0+1/\gamma}\right)^k}.$$

It is easy to see that if $\mathcal{R}_0>1$, $I_k\to(1-1/\mathcal{R}_0)\,N$, and if $\mathcal{R}_0<1$, $I_k\to 0$. Note that the discrete-time SIS model constructed with forward Euler's scheme is not a Beverton-Holt equation. We have proved the following results.

Theorem 3.1. *Consider the discrete SIS model (18) with $\phi>0$. If $\mathcal{R}_0<1$, all solutions approach the disease-free equilibrium $(N,0)$, and if $\mathcal{R}_0>1$, all solutions with positive initial conditions approach the endemic equilibrium $(N/\mathcal{R}_0,(1-1/\mathcal{R}_0)N)$. Specifically, if $\beta\neq\gamma$ and h sufficiently small so that $\phi=\frac{e^{(\beta-\gamma)h}-1}{\beta-\gamma\, e^{(\beta-\gamma)h}}>0$, then system (18) is an exact finite difference scheme for system (6).*

3.2. SIS **model with births and deaths (7).** Consider the SIS model (7) with births and deaths. The total population N is a constant. The basic reproduction number is $\mathcal{R}_0 = \beta/(\gamma+\mu)$. If $\mathcal{R}_0 < 1$, all solutions approach the disease-free equilibrium $(N, 0)$, and if $\mathcal{R}_0 > 1$, all solutions with positive initial conditions approach the endemic equilibrium $(N/\mathcal{R}_0, N(1-1/\mathcal{R}_0))$. The system can be reduced to the I equation: $I'(t) = \beta I(1 - I/N) - (\gamma+\mu)I$ with solution

$$I(t) = \frac{(\beta-\gamma-\mu)NI_0}{\beta I_0 + [(\beta-\gamma-\mu)N - \beta I_0]e^{-(\beta-\gamma-\mu)t}}$$

and an exact finite difference equation can be written as

$$\frac{I_{k+1}-I_k}{\phi} = \beta I_k\left(1 - \frac{I_{k+1}}{N}\right) - (\gamma+\mu)I_{k+1}, \text{ where } \phi = \frac{e^{(\beta-\gamma-\mu)h}-1}{(\beta-\gamma) - \mu\, e^{(\beta-\gamma-\mu)h}}.$$

Following the exact finite difference equation for I, we may construct an NSFD scheme as the following:

$$\begin{aligned} \frac{S_{k+1}-S_k}{\phi} &= \mu N - \frac{\beta}{N}S_{k+1}I_k - \mu S_{k+1} + \gamma I_{k+1}, \\ \frac{I_{k+1}-I_k}{\phi} &= \frac{\beta}{N}S_{k+1}I_k - (\mu+\gamma)I_{k+1}, \end{aligned} \tag{20}$$

with $\phi > 0$. If $\phi = \frac{e^{(\beta-\gamma-\mu)h}-1}{\beta-\gamma-\mu\, e^{(\beta-\gamma-\mu)h}}$, then (20) becomes an exact finite difference system. The total population for the discrete system is a constant. The conservative law $S_k + I_k = N$ holds for all $k > 0$ and $\phi > 0$. S_{k+1} and I_{k+1} can be solved explicitly in terms of S_k and I_k:

$$\begin{aligned} S_{k+1} &= \frac{N(S_k + \phi N(\mu+\gamma))}{N(1+\phi\mu+\phi\gamma) + \phi\beta I_k}, \\ I_{k+1} &= \frac{I_k N(1+\phi\beta)}{N(1+\phi\mu+\phi\gamma) + \phi\beta I_k}. \end{aligned} \tag{21}$$

The discrete SIS model (21) preserves positivity of solutions and the I equation is a Beverton-Holt equation with the solution

$$I_k = \frac{N(1-1/\mathcal{R}_0)}{1 + \left[\frac{N}{I_0}(1-1/\mathcal{R}_0) - 1\right]\left(\frac{\phi + 1/(\mu+\gamma)}{\phi\mathcal{R}_0 + 1/(\mu+\gamma)}\right)^k}.$$

It is easy to see that if $\mathcal{R}_0 > 1$, $I_k \to N(1-1/\mathcal{R}_0)$, and if $\mathcal{R}_0 < 1$, $I_k \to 0$. We have proved the following results.

Theorem 3.2. *Consider the discrete SIS model (20) with $\phi > 0$. If $\mathcal{R}_0 < 1$, all solutions approach the disease-free equilibrium $(N, 0)$, and if $\mathcal{R}_0 > 1$, all solutions with positive initial conditions approach the endemic equilibrium $(N/\mathcal{R}_0, (1-1/\mathcal{R}_0)N)$. Specifically, if $\beta - \gamma \neq \mu$ and h sufficiently small so that $\phi = \frac{e^{(\beta-\gamma-\mu)h}-1}{(\beta-\gamma)-\mu\, e^{(\beta-\gamma-\mu)h}} > 0$, system (20) is an exact finite difference scheme for system (7).*

3.3. SIS **model with immigration (8).** Consider the SIS model with immigration (8). The total population $N(t)$ satisfies $N'(t) = \lambda - \mu N(t)$. The basic reproductive number is $\mathcal{R}_0 = \beta/(\mu+\gamma)$. If $\mathcal{R}_0 < 1$, the disease-free equilibrium $\left(\frac{\lambda}{\mu}, 0\right)$ is globally asymptotically stable, and if $\mathcal{R}_0 > 1$, the endemic equilibrium

$\left(\frac{\lambda}{\mu\mathcal{R}_0}, \frac{\lambda}{\mu}(1-1/\mathcal{R}_0)\right)$ is stable. As $t \to \infty$, $I'(t) \to \beta I\left(1-\frac{\mu}{\lambda}I\right) - (\mu+\gamma)I$. Asymptotically, the solution paths behave as the solutions of the SIS model with births and deaths (7).

Similar to the NSFD SI system (15), we construct an NSFD scheme as the following:

$$\begin{aligned}\frac{S_{k+1}-S_k}{\phi} &= \lambda - \frac{\beta}{N_k}S_{k+1}I_k + \gamma I_{k+1} - \mu S_{k+1},\\ \frac{I_{k+1}-I_k}{\phi} &= \frac{\beta}{N_k}S_{k+1}I_k - \gamma I_{k+1} - \mu I_{k+1},\end{aligned} \tag{22}$$

with $\phi > 0$. The system satisfies the conservation law for N: $(N_{k+1}-N_k)/\phi = \lambda - \mu N_{k+1}$. If $N_0 \le \lambda/\mu$, N_k increases monotonically to λ/μ as $k \to \infty$ and if $N_0 \ge \lambda/\mu$, N_k decreases monotonically to λ/μ. S_{k+1} and I_{k+1} can be solved explicitly in terms of S_k, I_k, and N_k:

$$\begin{aligned}S_{k+1} &= \frac{(S_k+\phi\lambda)(1+\phi\mu)+\phi\gamma(\phi\lambda+N_k)}{(1+\phi\mu)(1+\phi\mu+\phi\gamma+\phi\beta I_k/N_k)},\\ I_{k+1} &= \frac{(1+\phi\mu+\phi\beta+\phi^2\beta\lambda/N_k)I_k}{(1+\phi\mu)(1+\phi\mu+\phi\gamma+\phi\beta I_k/N_k)}.\end{aligned}$$

The NSFD scheme (3.3) is an approximation to the SIS model with immigration (8). We are not able to find an exact finite difference scheme because there is no explicit solution to the original SIS model (8). The I equation of the above discrete system is not a Beverton-Holt equation. However, the asymptotic dynamics of I_k when $k \to \infty$, so $N_k \to \lambda/\mu$ is that of the Beverton-Holt and are the same as the I equation in (21). We have the following results.

Theorem 3.3. *If $\mathcal{R}_0 < 1$, the disease-free equilibrium $\left(\frac{\lambda}{\mu}, 0\right)$ of the discrete system (22) is globally asymptotically stable.*

Proof. We only need to show that I_k approaches 0 as $\mathcal{R}_0 < 1$. We will prove this in two cases: (i) $N_k \le \lambda/\mu$ and (ii) $N_k \ge \lambda/\mu$. If $N_k \le \lambda/\mu$, then from the I equation in (3.3), we have

$$I_{k+1} \le \frac{(1+\phi\mu+\phi\beta+\phi^2\beta\lambda\mu/\lambda)I_k}{(1+\phi\mu)(1+\phi\mu+\phi\gamma)} = \frac{1+\phi\beta}{1+\phi\mu+\phi\gamma}\cdot I_k. \tag{23}$$

If $N_k \ge \lambda/\mu$, then $N_{k+1} \le N_k$ and therefore $S_{k+1} = N_{k+1} - I_{k+1} \le N_k$. The I equation can also be written as

$$I_{k+1} = \frac{1+\phi\beta S_{k+1}/N_k}{1+\phi\mu+\phi\gamma}\cdot I_k \le \frac{1+\phi\beta}{1+\phi\mu+\phi\gamma}\cdot I_k.$$

In both cases, $I_k \to 0$ if $\mathcal{R}_0 < 1$. □

4. **Discussion.** Discrete-time SI and SIS models are constructed using nonstandard finite difference schemes. We have shown that the four NSFD schemes, (10), (13), (18), and (20), are exact finite difference schemes when the denominator function ϕ is chosen properly. With NSFD schemes, it is easier to construct discrete models that are dynamically consistent with the continuous models. In epidemic models, population variables are usually non-negative. The six NSFD schemes (10), (13), (15), (18), (20), and (22), reveal a simple rule to construct NSFD schemes—by preserving the positivity of solutions. In addition, we impose the condition that the conservation law holds on these discrete models. Surprisingly, the solutions of the

discrete models follow the solution paths of the continuous time models; a certain feasible region is positively invariant for both the continuous model and the discrete model. In other words, the discrete models are dynamically consistent with the continuous models.

The two types, SI and SIS, epidemic models are simple systems to test the NSFD schemes. We have shown that it is possible to design NSFD schemes that preserve almost all of the qualitative behavior of the continuous models. Since the positive variables are crucial to biology and population models, the simple ideas of designing a discrete model or a numerical scheme that preserves this particular property provides a simple and effective way to construct an NSFD scheme.

NSFD schemes are not fully understood. Further investigation is needed for more general 2-dimensional population models. In the future we would like to apply the same procedure to higher dimensional population or epidemic models and compare their results.

REFERENCES

[1] L.J.S. Allen, *Some discrete-time SI, SIR, and SIS epidemic models*, Math. Biosci., **124** (1994), 83-105.

[2] L.J.S. Allen, "An Introduction to Mathematical Biology," Prentice Hall, New Jersey, 2007.

[3] R. Anguelov and J.M.S. Lubuma, *Contribution to the mathematics of the nonstandard finite difference method and applications*, Numer. Methods Par. Diff. Equ., **17** (2001), 518–543.

[4] M. Chapwanya, Jean M.-S. Lubuma, and R.E. Mickens, *From enzyme kinetics to epidemiological models with Michaelis-Menten contact rate: Design of nonstandard finite difference schemes*, Computers and Mathematics with Applications, (2012), xxx–yyy.

[5] S. Elaydi, "An Introduction to Difference Equations," 3^{rd} edition, Springer, New York, 2005.

[6] H.W. Hethcote, *Qualitative analyses of communicable disease models*, Math. Biosci., **28** (1976), 335–356.

[7] H.W. Hethcote, *The Mathematics of Infectious Diseases*, SIAM Review, **42** (2000), 599–653.

[8] S.R.J. Jang, *Nonstandard finite difference methods and biological models*, in "Advances in the applications of nonstandard finite diffference schemes," edited by R.E. Mickens, World Scientific, New Jersey, 2005.

[9] P. Liu and S.N. Elaydi, *Discrete Competitive and Cooperative Models of Lotka-Volterra Type*, Journal of Computational Analysis and Applications, **3** (2001), 53–73.

[10] R.E. Mickens, "Nonstandard Finite Difference Models of Differential Equations," World Scientific, New Jersey, 1994.

[11] R.E. Mickens, "Advances in the applications of nonstandard finite difference schemes," World Scientific, New Jersey, 2005.

[12] R.E. Mickens, *Calculation of denominator functions for nonstandard finite difference schemes for differential equations satisfying a positivity condition*, Numerical Methods for Partial Differential Equations, **23** (2006), 672–691.

[13] R.E. Mickens, *Numerical integration of population models satisfying conservation laws: NSF methods*, Journal of Biological Dynamics, **1** (2007), 427–436.

[14] R.E. Mickens, *A SIR-model with square-root dynamics: An NSFD scheme*, Journal of Difference Equations and Applications, **16** (2010), 209–216.

[15] R.E. Mickens and T.M. Washington, *A note on and NSFD scheme for a mathematical model of respiratory virus transmission*, J. Difference Equations and Appl., **18** (2012), 525–529.

[16] L.-I.W. Roeger, *Nonstandard finite difference schemes for differential equations with* $\boldsymbol{n+1}$ *distinct fixed-points*, Journal of Difference Equations and Applications, **15** (2009), 133–151.

[17] L.-I.W. Roeger, *Dynamically consistent finite difference schemes for the differential equation* $dy/dt = b_n y^n + b_{n-1} y^{n-1} + \cdots + b_1 y + b_0$, Journal of Difference Equations and Applications, **18** (2012), 305–312.

[18] L.-I. W. Roeger and G. Lahodny, Jr., *Dynamically consistent discrete Lotka-Volterra competition systems*, Journal of Difference Equations and Applications, **19** (2013), 191–200.

Received August 2012; revised April 2013.

E-mail address: lih-ing.roeger@ttu.edu